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ANALYSIS AND CONSTRUCTION OF
EXTREMAL CIRCULANT AND OTHER
ABELIAN CAYLEY GRAPHS

ROBERT RODERICK LEWIS

Submitted for the degree of
Doctor of Philosophy in Mathematics
The Open University
Milton Keynes, UK

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Abstract

This thesis concerns the analysis and construction of extremal circulant and other Abelian Cayley graphs. For the purpose of this thesis, extremal graphs are understood as graphs with largest possible order for given degree and diameter, and the search for them is called the degree-diameter problem. The emphasis is on circulant graphs and on families of graphs defined for infinite diameter classes for given fixed degrees.

Most studies in the degree-diameter problem have employed candidate algebraic structures to generate graphs that successively improve on previous best results. In contrast, this study has made extensive use of computer searches to find extremal graphs and graphs families directly, and has then sought the algebra that describes them. In this way, the maximum degree for which largest-known circulant graph families have been discovered, with order greater than the legacy lower bound, has been increased from 7 to 20 and beyond.

Topics covered include graphs in the following categories, undirected unless stated otherwise: circulant, other Abelian Cayley, bipartite circulant, arc-transitive circulant, directed circulant and mixed circulant; and their main properties such as distance partition, odd girth and automorphism group size.

A major aspect of this thesis is the analysis of a matrix associated with each graph family, the *lattice generator matrix*, with newly discovered properties such as quasimaximality, radius maximality and eccentricity. Important new relationships between graph families of common dimension have also been discovered: translation, conjugation and transposition.

An Extremal Order Conjecture is established for extremal undirected circulant and other Abelian Cayley graphs of any degree and diameter. An equivalent conjecture for directed circulant graphs and certain classes of mixed circulants is also established. Most of the extremal and largest-known graphs and graph families presented here have been discovered by the author and are documented comprehensively in the appendices.

PREFACE

Everything about us, everything around us, everything we know and can know of is composed ultimately of patterns of nothing; that's the bottom line, the final truth. So where we find we have any control over those patterns, why not make the most elegant ones, the most enjoyable and good ones, in our own terms?

Iain M. Banks, in *Consider Phlebas*.

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First of all I would like to thank my principal supervisor Professor Jozef Širáň for introducing me to the degree-diameter problem for circulant graphs, for permitting me the freedom to explore a relatively narrow field of research in some depth, and for his encouragement and guidance during the past seven years.

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I would like to acknowledge the support of the Open University STEM computing department for the provision of extensive computing resource for the execution of my search programs.

Special thanks are due to Grahame Erskine, who has been a close colleague throughout my PhD, providing practical assistance with computing software and systems, offering insightful challenge and support, and collaborating on occasional pieces of research.

And finally, I offer my most grateful thanks to my wife Catriona Portefaix for her encouragement of my progress, her interest in my discoveries, and also for practical support in formatting and typing up this thesis in its many revisions.

Declaration

Unless otherwise stated, the results presented in this thesis are my own work. Some of them have already been published as papers and in my MSc dissertation, and the text of the relevant sections has been adapted from those papers where appropriate.

The complete versions of some of the proofs are extremely long because of the number of cases and the number of exceptions. For such proofs, only a representative example is included here, with a reference to where the full set may be found.

The majority of the graphs and graph families presented here are my own discoveries. These are specified at the head of each of the seven appendices.

Rob Lewis

March 2021

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CHAPTER 1

INTRODUCTION

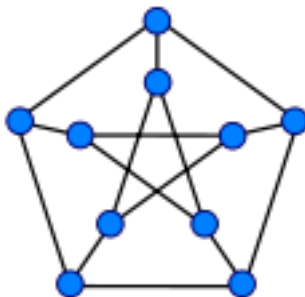
This thesis presents the results of an extensive study of the degree-diameter problem for Abelian Cayley graphs and graph families. The initial work was on undirected circulant graphs. This was subsequently extended to include some specific categories of circulant graphs: bipartite, arc-transitive, directed and mixed. The search was also widened to address all undirected Abelian Cayley graphs.

This chapter introduces the subject and some necessary concepts, definitions and notation. It concludes with a synopsis of the rest of the thesis.

1.1 The degree-diameter problem

The goal of the *degree-diameter problem* is to find graphs with the largest possible number of vertices for a given maximum degree and diameter. In this thesis, such graphs are called *extremal* graphs. For example, the extremal graph of maximum degree 3 and diameter 2 is the Petersen graph, with order 10, see Figure 1.1.

Figure 1.1: Petersen graph



Despite the simplicity of the question, the degree-diameter problem is a difficult combinatorial problem. For the general case of undirected graphs, according to Combinatorics Wiki [7], only seven graphs with degree greater than 2 are confirmed to be extremal. The largest of these is the Hoffman-Singleton graph, see Table 1.1.

Table 1.1: Order of confirmed extremal graphs of degree greater than 2

Degree	Diameter		
	2	3	4
3	10*	20	38
4	15	-	-
5	24	-	-
6	32	-	-
7	50 [†]	-	-

* Petersen graph

[†] Hoffman-Singleton graph

The order of any graph of maximum degree $d \geq 2$ and diameter k is bounded above by the *Moore bound* [20]:

$$M(d, k) = \begin{cases} d^{\frac{(d-1)^k - 1}{d-2}} & \text{if } d > 2 \\ 2k + 1 & \text{if } d = 2. \end{cases}$$

A graph with order equal to the Moore bound is called a *Moore graph*. According to the Hoffman-Singleton Theorem [20], for maximum degree $d > 2$ and diameter $k > 1$, the only possible Moore graphs have diameter $k = 2$ and degree $d = 3, 7$ or 57 . For degree 3, this is the Petersen graph of order 10; for degree 7, the Hoffman-Singleton graph of order 50; but for degree 57, the existence of a Moore graph of order 3250 is unknown. The existence or non-existence of the Moore graph of degree 57, diameter 2 and order 3250 is probably the most famous open question in the degree-diameter problem.

Within this context it is useful to clarify precisely what is meant in this thesis by the *existence* of a graph or family. For the degree-diameter problem, the diameter of a graph is considered to be an intrinsic element of its specification, alongside its degree and order. This is especially true for an Abelian Cayley graph family, where for any given degree, the order $n(k)$ and generating set $\{g_1(k), \dots, g_f(k)\}$ of any graph in the family may be specified by polynomials in the diameter k . Therefore *existence* is understood to refer to graphs with given combinations of degree, diameter, order and, optionally, generating set. As we shall see later, for Abelian Cayley graphs of degree $2f$ and diameter k , their existence is equivalent to the existence of the corresponding lattice covering of \mathbb{Z}^f by Lee spheres of radius k .

Definition 1.1. An Abelian Cayley graph of given degree, diameter, order and, optionally, generating set is said to *exist* if the graph with specified degree, order and generating set also has the specified diameter. Similarly, an Abelian Cayley graph family is defined to *exist* if the value of the diameter k that is the independent

variable in the polynomials defining the order and generating set of any graph in the family is equal to the diameter of the graph.

From the literature, it is seen that the degree-diameter problem has been tackled for undirected, directed and mixed graphs. In addition to the general case, various subproblems have also been defined by restricting the scope to specific graph classes including vertex-transitive graphs and Cayley graphs. A general background on the degree-diameter problem is presented in the comprehensive survey by Miller and Širáň [35] and the tables of extremal and largest-known graphs on the CombinatoricsWiki website [7].

Only in relatively few cases have the largest-known graphs been proved to be extremal, typically restricted to degree 3 for small diameter or diameter 2 for small degree. Undirected circulant graphs, being highly structured, are a noteworthy exception, with extremal graphs of degrees 2, 3, 4 and 5 confirmed for all diameters, and largest-known graphs of degree 6 and 7 that are conjectured extremal. Prior to the author's work, the only other circulant graphs with order greater than a legacy lower bound were degree 8 for all diameters (since superseded by the author) and some higher degree graphs of diameters 2 and 3. See Table 1.2 for the order of these graphs up to diameter 10.

Table 1.2: Order of extremal and largest-known circulant graphs up to diameter 10 (excluding legacy lower bound): 2013 status

Degree	Diameter								
	2	3	4	5	6	7	8	9	10
2	5	7	9	11	13	15	17	19	21
3	8	12	16	20	24	28	32	36	40
4	13	25	41	61	85	113	145	181	221
5	16	36	64	100	144	196	256	324	400
6	21	55	117	203	333	515	737	1027	1393
7	26	76	160	308	536	828	1232	1764	2392
8	35	104	241	511	967	1681	2737	4231	6271
9	42	130							
10	51	177							
11	56								
12	67								
13	80								
14	90								
15	96								
16	112								
17									
18									
19									
20									

As a result of the work presented in this thesis, circulant graph families with order greater than the lower bound have been discovered up to degree 20 and beyond, for all diameters. Some are conjectured extremal; all are largest known. Similar families have also been discovered for the wider class of Abelian Cayley graphs, up to degree 15. In Table 1.3, the current status of extremal and largest-known circulant graphs is presented.

Table 1.3: Order of extremal and largest-known circulant graphs up to degree 20 and diameter 10: 2021 status

Degree	Diameter								
	2	3	4	5	6	7	8	9	10
2	5	7	9	11	13	15	17	19	21
3	8	12	16	20	24	28	32	36	40
4	13	25	41	61	85	113	145	181	221
5	16	36	64	100	144	196	256	324	400
6	21	55	117	203	333	515	737	1027	1393
7	26	76	160	308	536	828	1232	1764	2392
8	35	104	248	528	984	1712	2768	4280	6320
9	42	130	320	700	1416	2548	4304	6804	10320
10	51	177	457	1099	2380	4551	8288	14099	22805
11	56	210	576	1428	3200	6652	12416	21572	35880
12	67	275	819	2120	5044	10777	21384	39996	69965
13	80	312	970	2676	6256	14740	30760	57396	106120
14	90	381	1229	3695	9800	23304	49757	103380	196689
15	96	448	1420	4292	12232	32092	68944	142516	276928
16	112	518	1788	5847	17733	45900	107748	232245	479255
17	130	570	1954	6468	20360	57684	136512	321780	659464
18	138	655	2645	8425	27273	80940	208872	492776	1078280
19	156	722	2696	9652	31440	99420	258040	652004	1416256
20	171	815	3175	12396	42252	132720	371400	930184	2232648

Graphs in bold were discovered by the author

The symmetry and simplicity of circulant graphs, as Cayley graphs of cyclic groups, admit to their representation by polynomials in the diameter or the degree, which has been an important factor in the advances achieved so far. These polynomials define the vertex and edge sets of families of such graphs.

1.2 Graph definitions

For a general introduction to graph theory and the standard definitions, see Godsil and Royle's excellent book 'Algebraic graph theory' [16] or similar.

A *graph* is a set of *vertices* together with a set of *edges*. Each *edge* is defined by a pair of vertices. If the pair is unordered then the edge is *undirected*. If the pair is ordered then the edge is *directed* and is called an *arc*. An edge may be considered as a pair of oppositely directed arcs. A graph with only undirected edges is called an *undirected*

graph. A graph with only arcs is called a *directed* graph, or *digraph* for short. A graph with at least one undirected edge and one arc is called a *mixed* graph. The *order* of a graph is its number of vertices and the *size* of the graph, its number of edges. A *simple* graph is a graph in which each edge connects two distinct vertices and no two edges connect the same pair of vertices. A *walk* is a sequence of edges or arcs joining a sequence of vertices. A *path* is a walk where no vertex is repeated, except for the first and last in case of a closed path. A closed path is called a *cycle*. The *length* of a path is the number of edges or arcs in the path. The *distance* between two vertices is the length of the shortest path between them. The *diameter* of a graph is the largest distance between any two vertices. The *girth* of a graph is the length of its shortest cycle. The *odd girth* of a graph is the length of its shortest odd-length cycle.

A *connected* graph is one in which each pair of vertices forms the endpoints of a path. The *degree* of a vertex is its number of incident edges, also sometimes called the *valency*. For directed and mixed graphs, the *indegree* and *outdegree* are similarly defined. The *maximum degree* of a graph is the maximum degree of its vertices. A graph is *regular* if all its vertices have the same degree. A graph is *finite* if it has a finite number of vertices. In this thesis, we only consider circulant and other Abelian Cayley graphs, which are all finite, simple, connected and regular. Unless otherwise stated, they are undirected.

A graph is *vertex transitive* if, for any two vertices in the graph, there is an automorphism mapping the first to the second. A graph is *edge transitive* if, for any two edges in the graph, there is an automorphism mapping the first to the second. A graph is *arc transitive* if, for any two arcs in the graph, there is an automorphism mapping the first to the second. This definition includes edges, which may be considered in either orientation. Clearly, any arc-transitive graph is vertex transitive and edge transitive.

1.3 Abelian Cayley graphs and graph families

A *Cayley graph* $X(A, C)$ is defined for a group A and a *connection set* $C \subset A \setminus \{e\}$, where e is the identity element of the group and C is a generating set of A , as follows. The vertices of X are identified with the elements of A . The edge set of X consists of the arcs (u, uc) for any $u \in A$ and $c \in C$. If C is symmetric ($C = C^{-1}$) then X is an undirected graph. By definition, every Cayley graph is regular, with the degree d of each vertex equal to the cardinality of C , and also vertex transitive. An *Abelian Cayley graph* is a Cayley graph $X(A, C)$ where the group A is Abelian. In the particular case where A is a cyclic group of order n , then $A \cong \mathbb{Z}_n$ and the graph

$X(\mathbb{Z}_n, C)$ is called a *circulant graph*. The group \mathbb{Z}_n has at most one involution, $n/2$, and only in case n is even. All other elements of \mathbb{Z}_n belong to complementary pairs $\pm i$ or equivalently i and $n - i$. Any calculations involving the values of elements of a cyclic group \mathbb{Z}_n of order n will assume the arithmetic is modulo n , sometimes denoting $n - i$ by $-i$.

Any finite Abelian group is isomorphic to the direct product of a family of cyclic groups. In this context, it is convenient to introduce a new term, *cyclic rank*.

Definition 1.2. The *cyclic rank* of a finite Abelian group is the minimum number of cyclic subgroups whose direct product is isomorphic to the group.

For example, the group $A = \mathbb{Z}_{16} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ has cyclic rank 3 and cyclic orders 16, 4 and 4. The cyclic rank of an Abelian Cayley graph is defined to be the cyclic rank of the corresponding Abelian group. Therefore, a circulant graph has cyclic rank 1.

By definition, any undirected circulant graph on n vertices has rotational and reflexive symmetries, so that its automorphism group is either the dihedral group on n points, D_n , of order $2n$, or contains the dihedral group as a subgroup. Similarly, given that any Abelian group is isomorphic to a direct product of cyclic groups, the automorphism group of an undirected Abelian Cayley graph will include as subgroups dihedral groups relating to each cyclic component. Therefore, the order of the automorphism group of an undirected Abelian Cayley graph of order n will be a multiple of $2n$. We define this multiple to be the *dihedral index* of the automorphism group, often abbreviated to DI.

In the literature the symbol d is variously used to denote the degree or the diameter or the dimension of the graph. Adopting the terminology of Macbeth, Šiagiová and Širáň [32], we will use d for degree and k for diameter.

For an undirected circulant graph $X(\mathbb{Z}_n, C)$, if n is odd, $\mathbb{Z}_n \setminus \{0\}$ has no element of order 2. Therefore, C has even cardinality, say $d = 2f$, and comprises f complementary pairs of elements, with one of each pair strictly between 0 and $n/2$. Any set of size f containing exactly one element from each pair is sufficient to uniquely determine the connection set and is called a *generating set*. Without loss of generality, we will usually choose the generating set which is comprised of the f elements of C between 0 and $n/2$ as the canonical generating set G for X . However, it should be noted that replacing any generator by its additive inverse has no impact on the connection set and therefore leaves the graph unchanged.

If n is even, $\mathbb{Z}_n \setminus \{0\}$ has just one element of order 2, namely $n/2$. In this case, C comprises f complementary pairs of elements, as for odd n , with or without the addition of the involutory element $n/2$. If C has odd cardinality, so that $d = 2f + 1$, then the value of its involutory element is defined by the value of n . Therefore, for a circulant graph of given order and degree, its connection set C is completely defined by specifying its generating set G . The cardinality of the connection set is equal to the degree d of the graph, and the cardinality of the generating set, f , is defined to be the *dimension* of the graph.

Clearly, if every element of a generating set is multiplied by a constant factor that is co-prime with the order of the graph, then the resultant set will also be a generating set of a graph which is isomorphic to the first. Therefore, the isomorphism class of a circulant graph could have a number of different generating sets. Not all isomorphism classes of circulant graphs have a *primitive* generating set (where one of the generators is 1). An example of an extremal circulant graph with no primitive generating set is the graph with degree 9, diameter 2, order 42 and generating set $\{2, 7, 8, 10\}$ along with the involution, 21.

We now present two important formal definitions: *diameter class* and *Abelian Cayley graph family*. These two concepts are fundamental to most of the results presented in this thesis. It is a consequence of the two principles discussed later in Section 2.2, concerning an equivalence between Abelian Cayley graphs of dimension f and lattice coverings of \mathbb{Z}^f by Lee spheres, that any Abelian Cayley graph of degree d and diameter k has a special relationship with other degree d graphs with diameters that belong to a regular arithmetic sequence including k . The order of each of these graphs is defined by a common polynomial in the diameter of degree f . The set of all graphs related in this way is called a family, and the set of diameters for which they exist is called a diameter class.

Definition 1.3. For a given dimension f , a *diameter class* K is a subset of a residue class such that $K = \{k^* + nh : n \in \mathbb{N}\}$, where h is a constant multiple of $f/2$ if f is even and otherwise a multiple of f . The *root* of a diameter class is its lowest member k^* , and its *period* is nh . Such a diameter class is usually referred to by its root, k^* , in which case it is assumed that $h = f/2$ if f is even and $h = f$ if f is odd, unless otherwise stated. For each degree, we also define the *principal diameter class* to be the class containing f for odd degree or $(f - 1)/2$ for even degree.

Definition 1.4. For a given degree d and corresponding dimension $f = \lfloor d/2 \rfloor$, and for a given diameter class K , an *Abelian Cayley graph family* \mathcal{X}_d is an infinite set of graphs $\mathcal{X}_d = \{X_d(k) : k \in K\}$ where $X_d(k)$ is an Abelian Cayley graph of degree d , diameter k , order $n(k)$ and generating set $\{g_1(k), \dots, g_f(k)\}$, where $n(k)$ and $g_i(k)$ are

polynomials in k of maximum degree f . A *circulant graph family* is similarly defined. Conversely, a family may also be defined for a given fixed diameter and an infinite regular sequence of degrees.

For the degree-diameter problem, there are four important properties related to graph order: *upper bound*, *extremal*, *largest known* and *lower bound*.

Definition 1.5. An *upper bound* is a value that is greater than or equal to the order of any graph of its class for a given degree and diameter k . For various classes of graph this is denoted as follows:

- $Upp_{circ}(d, k)$ Circulant graphs of degree d and diameter k
- $Upp_{circ}^{bip}(d, k)$ Bipartite circulant graphs of degree d
- $Upp_{circ}^{dir}(z, k)$ Directed circulant graphs of directed degree z
- $Upp_{circ}^{mix}(z, d, k)$ Mixed circulant graphs of directed degree z and undirected degree d
- $Upp_{AbCay}(d, k)$ Abelian Cayley graphs of degree d

Definition 1.6. An *extremal* graph has order that is greater than or equal to the order of any graph of its class for a given degree and diameter k . In most cases, the extremal graphs are unknown and are yet to be discovered. Such a graph has *extremal* order, denoted as follows:

- $Ext_{circ}(d, k)$ Circulant graphs of degree d and diameter k
(variants as for upper bound cases)

Definition 1.7. A *largest-known* graph has order that is greater than or equal to the order of any known graph of its class for a given degree and diameter k . A largest-known graph might or might not be extremal. It may be verified to be extremal by mathematical proof or by exhaustive search for all possible graphs of greater order up to an upper bound. Such a graph has *largest-known* order, denoted as follows:

- $LK_{circ}(d, k)$ Circulant graphs of degree d and diameter k
(variants as for upper bound cases)

Definition 1.8. A *lower bound* is a value that is less than or equal to the order of an extremal graph of the same class for a given degree and diameter k .

- $Low_{circ}(d, k)$ Circulant graphs of degree d and diameter k
(variants as for upper bound cases)

These definitions can also be extended to families of graphs of a given type, for fixed degree and any diameter in a given diameter class, or fixed diameter and any degree in a given degree class.

1.4 Vector spaces and norms

In this thesis we consider two vector spaces of arbitrary dimension f . One is the standard Euclidean space of dimension f , \mathbb{R}^f , where \mathbb{R} denotes the real numbers, with the Euclidean norm defined by the inner product of a vector. The other is an f -fold Cartesian product of the set of integers, \mathbb{Z}^f . With addition of vectors defined element-wise, this is isomorphic to the free Abelian group on f generators. We will usually apply the Manhattan norm to \mathbb{Z}^f instead of the Euclidean norm.

Definition 1.9. Given any dimension f and any vector $\mathbf{v} = (v_1, \dots, v_f) \in \mathbb{Z}^f$, the *Manhattan norm (or length)* is defined by $\delta(\mathbf{v}) = \sum_{i=1}^f |v_i|$ for the length of the vector \mathbf{v} , and then the derived Manhattan distance by $\delta(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^f |u_i - v_i|$ for the distance between vectors \mathbf{u} and \mathbf{v} .

Definition 1.10. For positive integers f, k , we define the f -dimensional *Lee sphere* of radius k , $S_{f,k}$ to be the set of elements of \mathbb{Z}^f which can be expressed as a word of length at most k in the canonical generators \mathbf{e}_i of \mathbb{Z}^f , taken positive or negative. Equivalently, $S_{f,k}$ is the set of points in \mathbb{Z}^f distant at most k from the origin under the Manhattan norm: $S_{f,k} = \{(x_1, \dots, x_f) \in \mathbb{Z}^f : |x_1| + \dots + |x_f| \leq k\}$. Although called a sphere, it appears more diamond-like than spherical, having the approximate form of a regular dual f -cube.

1.5 Vector notation for polynomials

As mentioned, throughout this thesis, with the exception of Chapter 14, graph families are defined for a fixed degree, with variable diameter specified by a diameter class. Thus, their order polynomials are also expressed in terms of the variable k , their diameter. In order to facilitate an understanding of the structure of Abelian Cayley graph families of given degree d and dimension $f = \lfloor d/2 \rfloor$, we will find it extremely useful to convert the polynomials of maximum degree f in the diameter for graph order and generators into polynomials in a related variable. The first step is to replace k with the term $2a$ where $a = 2k/f$. However, this variable a is in general only integral for a single diameter class $K = \{k : k \equiv 0 \pmod{f}\}$. In order to ensure the new variable remains integral for each diameter class K , we introduce a constant c_K such that $a = (2k + c_K)/f$ is integral for all $k \in K$. If K has root k^* , then an admissible value for c_K is given by $c_K = 2(f - k^*) \pmod{f}$. In order to ensure $-f/2 \leq c_K < f/2$, we refine the expression to give $c_K = (2(f - k^*) + \lfloor f/2 \rfloor) \pmod{f - \lfloor f/2 \rfloor}$. For clarity, in each case, a will always be explicitly defined.

A polynomial of degree f in k , $P'(k)$, is thus transformed into a polynomial of degree f in $2a$, $P(2a)$. Finally, we adopt a standard vector notation for $P(2a)$ to streamline presentation.

Notation 1.11. For the polynomial of degree f , $P(2a) = \sum_{i=0}^f c_i(2a)^i$, the corresponding vector notation is $P(2a) = (c_f \ c_{f-1} \ \dots \ c_0)$. Moreover, to divide each coefficient by a common denominator b , we write $(c_f \ c_{f-1} \ \dots \ c_0)/b$ for $P(2a)/b$. Note that the vector notation represents a polynomial in $2a$ and not a .

1.6 Notation for extremal and largest-known graphs and families

Throughout this thesis, and especially in the appendices, extremal and largest-known graph families and individual graphs are identified uniquely by a short code, such as F6:2B. The initial letter refers to the category of graph or family and is followed by the degree. For a graph family, the number after the colon is its diameter class; for individual graphs, it is its specific diameter. Any final letter indicates its isomorphism class (if there is more than one). Lower case letters ‘a’ and ‘b’ refer to distinct transpose pairs; upper case letters ‘A’, ‘B’, ... refer to isomorphism classes that are unrelated by transposition. The tables below list all the categories of graphs and families and give an example of each.

Table 1.4: Code for extremal and largest-known graph families

Category of family	Appendix	Initial letter	Code example	Degree*		Diameter class		Iso class
				dir	undir		(mod)	
Circulant	A	F	F6:2B	-	6	2	3	B
Bipartite circulant	B	D	D11:4a	-	11	4	5	a
Abelian Cayley	C	A	A14:0	-	14	0	7	-
Directed circulant	G	H	H2:2B	2	-	2	3	B
Mixed circulant	G	M	M2-1:1A	2	1	1	3	A

* directed, undirected

Table 1.5: Code for extremal and largest-known graphs that are not in largest-known families

Category of graph	Appendix	Initial letter	Code example	Degree	Diameter*	Isomorphism class
Circulant	D	G	G6:2C	6	2	C
Bipartite circulant	E	E	E11:4	11	4	-
Abelian Cayley	F	B	B14:2	14	2	-

* specific diameter, not diameter class

1.7 Computing strategies, programs and resources

Searches for circulant and Abelian Cayley graphs of a given degree were conducted using purpose-built computer programs written in either C++ or GAP programming languages. Two strategies were followed.

The first strategy requires the degree, diameter and order to be specified. Then a program, written in C++, iterates through all possible combinations of generators to check whether a graph with the specified parameters exists. The program can be set to stop as soon as it finds the first such graph or to continue to find all such graphs. If the program stops without finding a graph, then this indicates that no such graph exists. In order to verify that a largest-known graph of a given degree and diameter is actually extremal, this program can be run for all orders greater than the largest-known order up to an upper bound. For small degree and diameter, this is quickly achieved. But as the degree or diameter increases, the time taken increases exponentially. By degree 10 and diameter 10 the program can run for many days to check a single order, and above degree 11 is no longer feasible.

The second strategy was developed to take advantage of the discovery of the relationship between graph families (containing a graph for each diameter in a specified diameter class) and lattice generator matrices in a specified canonical format. It requires the degree and diameter class to be specified, but not the order. This strategy is exponentially more efficient than the first strategy for two reasons: the search space of admissible matrices is much smaller than the set of admissible generator sets, and the approach discovers complete graph families rather than individual graphs. These programs are also written in C++. There is a restriction that the search only finds graph families that are quasimaximal. We will see later that it is conjectured that extremal graph families of all degrees are quasimaximal. However if a graph family had greater order, then this search method would not find it. The discovery of the largest-known circulant graph families of degree 12 and above depended on adopting this second strategy as replacement of the first. For both strategies, extensive use was made of the Open University STEM computer cluster.

As a check on the existence of a graph that has been identified by either strategy, a different program, also written in C++ but using different logic developed independently by a colleague, is subsequently run. With this program, the degree, order and generating set are specified and the program calculates the diameter of the graph. This provides independent verification of the existence of a graph with the specified parameters: degree, diameter, order and generating set.

A third step is then adopted for candidate graphs of order less than 100,000. A program developed in GAP (a system for computational discrete algebra with particular emphasis on computational group theory) is run as a further verification on the existence of the graph and to determine its automorphism group.

The existence of every graph of order less than 100,000 documented in the appendices has been triple-verified in this way.

1.8 Synopsis of the chapters

The subject of each chapter is described below in a brief paragraph. These are repeated at the start of each chapter for convenience.

Chapter 2 describes the history and the state of knowledge of the degree-diameter problem for circulant graphs prior to the work for this thesis. This includes an equivalence between undirected Abelian Cayley graphs of arbitrary dimension f and lattice coverings of \mathbb{Z}^f , extremal and largest-known undirected circulant graph families up to degree 7, extremal directed circulant graphs of degree 2, and legacy lower and upper bounds for the order of extremal Abelian Cayley graphs.

In Chapter 3, we present an Extremal Order Conjecture for undirected circulant and Abelian Cayley graphs. The newly discovered graph families described in subsequent chapters are all evaluated against this conjecture. Any graph families with order polynomial as specified for the conjecture are called quasimaximal.

In Chapter 4, newly discovered largest-known circulant graph families up to degree 11 are described, along with a proof of the existence of the degree 8 families and lattice generating vectors for the degree 10 proof. The degree 8 and 9 families and the degree 8 existence proof were previously the subject of the author's MSc dissertation.

Chapter 5 discusses some important properties of Abelian Cayley graph families: maximal distance partition levels, distance partition profiles, quasimaximality and maximum odd girth. Two essential relationships between graph families are also presented: conjugation, which relates two quasimaximal families of the same degree, and translation, between an odd-degree family and an even-degree family of the same odd dimension.

In Chapter 6, we define the canonical lattice generator matrix (LGM) of an Abelian Cayley graph family. Some interesting properties and relations are discussed for both quasimaximal and subquasimaximal graph families: radius maximality and eccentricity. An important theorem is established that proves the existence of all

graphs in an Abelian Cayley graph family given the existence of graphs of low diameter. The equivalence is established of a graph family being quasimaximal, its graphs having maximum odd girth and its canonical LGM being radius maximal. In the final section, the graph family relationships of translation, conjugation and transposition are defined in terms of their canonical LGMs.

Chapter 7 describes the enumeration of all the quasimaximal degree 7 circulant graphs of diameter class $0 \pmod{3}$ by establishing a bijection with the set of all matrices in canonical quasimaximal LGM format.

Chapter 8 presents newly discovered largest-known circulant graph families up to degree 20, all quasimaximal. The efficient search for these families depended on LGM properties discussed in Chapter 6. Many of these families are related to others by the relationships described in Chapter 5. The interaction of the three relationships - translation, conjugation and transposition - for the largest-known circulant graph families of each dimension is presented graphically within a *dimensional frame*.

In Chapter 9, largest-known bipartite circulant graph families up to degree 11 are presented, along with some theorems establishing how bipartite circulant graph families of any degree are related to corresponding non-bipartite families. The bipartite/non-bipartite relationship, alongside the three previously discussed relationships of translation, conjugation and transposition are presented graphically for each dimension by a dimensional frame. An extremal order conjecture for bipartite circulant graph families is presented.

Chapter 10 presents largest-known non-circulant Abelian Cayley graph families up to degree 15, and one of degree 19, all quasimaximal.

Chapter 11 illustrates a surprising and beautiful relationship between Lucas polynomials and an infinite sequence of quasimaximal circulant graph families that are arc-transitive and have multiplicative generating sets. For any dimension, the order and generating sets of these families are defined in terms of Lucas polynomials.

Chapter 12 describes some series of circulant graph families beyond degree 20, created by extending sets of LGMs with common formats to higher dimensions. These families are conjectured to exist for all dimensions.

In Chapter 13, the Extremal Order Conjecture is extended to the third coefficient in the order polynomial. A conjecture is also discussed that all extremal Abelian Cayley graphs above threshold diameters are members of quasimaximal families. Some established theorems from the literature on asymptotically low-density lattice

coverings are considered to investigate whether they might indicate the existence, for sufficiently large dimension, of extremal Abelian Cayley graph families of order greater than determined by the Extremal Order Conjecture. However, the validity of these theorems is questioned, and the conjecture is considered to remain valid.

In contrast to all the other chapters, in Chapter 14, graph families are considered where the diameter is fixed instead of the degree. Graph families with diameter 2 and arbitrary degree are discussed. Some improved lower and upper bounds are established for their extremal order.

Chapter 15 is the first chapter where the graphs are not undirected. Directed and mixed circulant graphs and graph families are explored for small fixed directed and undirected degree, building on the legacy position described in Section 2.6. As defined in Section 1.2, a directed graph is a graph where all the edges are directed edges, called arcs. A mixed graph is a graph with at least one undirected edge and one arc. Directed and mixed Abelian Cayley graphs have a connection set that is not inverse-closed. In contrast to undirected circulant graphs, it appears that extremal directed and mixed circulant graphs of dimension 3 and above do not belong to graph families with regular diameter classes. An extremal order conjecture is presented for directed and mixed circulant graphs with undirected degree below 4.

Chapter 16 contains the conclusion, with a brief review of each chapter along with potential avenues for further research.

After the bibliography, there is an extensive set of appendices of graphs and graph families, serving not only as evidence for the work accomplished for this thesis, but also as a reference document for future researchers.

CHAPTER 2

BACKGROUND

Chapter 2 describes the history and the state of knowledge of the degree-diameter problem for circulant graphs prior to the work for this thesis. This includes an equivalence between undirected Abelian Cayley graphs of arbitrary dimension f and lattice coverings of \mathbb{Z}^f , extremal and largest-known undirected circulant graph families up to degree 7, extremal directed circulant graphs of degree 2, and legacy lower and upper bounds for the order of extremal Abelian Cayley graphs.

2.1 Undirected Abelian Cayley graphs and lattice coverings of \mathbb{Z}^f

Much of the work for this thesis depends on two important principles that are stated below as Propositions 2.1 and 2.2 and discussed later. Let A be a finite Abelian group generated by $\{g_1, \dots, g_f\}$. Let X be the undirected Abelian Cayley graph of A with generating set $\{g_1, \dots, g_f\}$. Let \mathbf{e}_i be the canonical generators of the free Abelian group on f generators, \mathbb{Z}^f , where \mathbf{e}_i is the i -th unit basis vector of \mathbb{Z}^f . Let $S_{f,k}$ be the Lee sphere of dimension f and radius k (Manhattan norm).

Proposition 2.1. *There is a unique canonical epimorphism from \mathbb{Z}^f onto A sending \mathbf{e}_i to g_i for all i . The kernel of the epimorphism, L_f , is a lattice in \mathbb{Z}^f with corresponding lattice vectors $\mathbf{v}_1, \dots, \mathbf{v}_f$. Then A is isomorphic to \mathbb{Z}^f/L_f , and the Cayley graph of A with given generators is isomorphic to the Cayley graph of \mathbb{Z}^f/L_f with generators being the cosets $\mathbf{e}_i L_f$ for $1 \leq i \leq f$.*

Proposition 2.2. *(Dougherty and Faber [10]). The graph X has diameter at most k if and only if $S_{f,k} + L_f = \mathbb{Z}^f$, so that this is a lattice covering of \mathbb{Z}^f .*

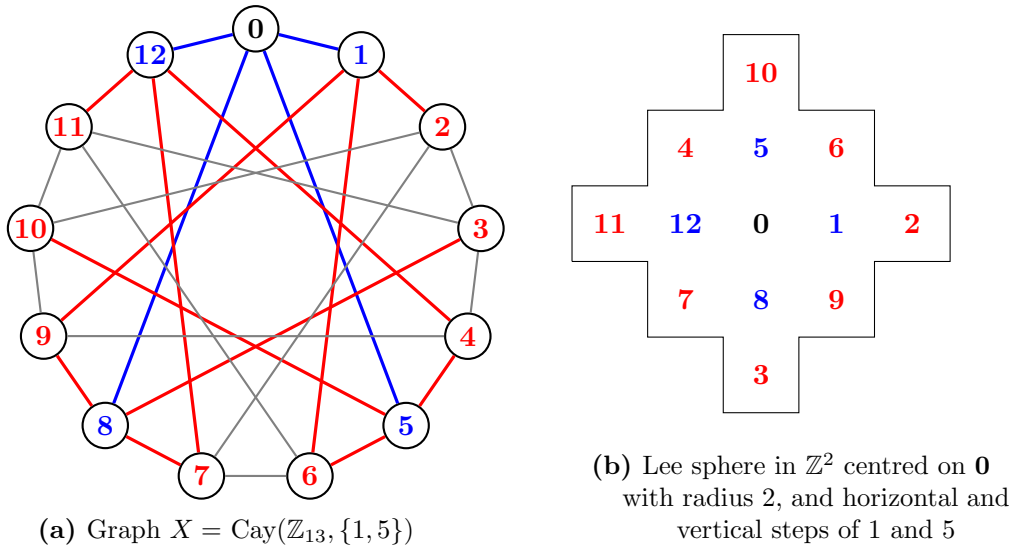
Proposition 2.1 establishes a relationship between finite Abelian groups with f generators and the free Abelian group on f generators, isomorphic to \mathbb{Z}^f , that greatly facilitates study of the degree-diameter problem for Abelian Cayley graphs. In case one of the generators g_i is an involution, so that the graph X has odd degree, then there is an alternative version of Proposition 2.2 stated later as Proposition 2.5.

These principles were the basis of Dougherty and Faber's existence proofs of the largest-known degree 6 and 7 circulant graph families, [10]. They also offer an efficient

approach to the search for families of higher dimension, as we shall see in later chapters.

This relationship is illustrated in Figure 2.1 for the simple case of a circulant graph $X = \text{Cay}(\mathbb{Z}_{13}, \{1, 5\})$, with degree $d = 4$, dimension $f = 2$, diameter $k = 2$, order $n = 13$ and generating set $\{1, 5\}$, so that its connection set is $\{1, 5, 8, 12\}$.

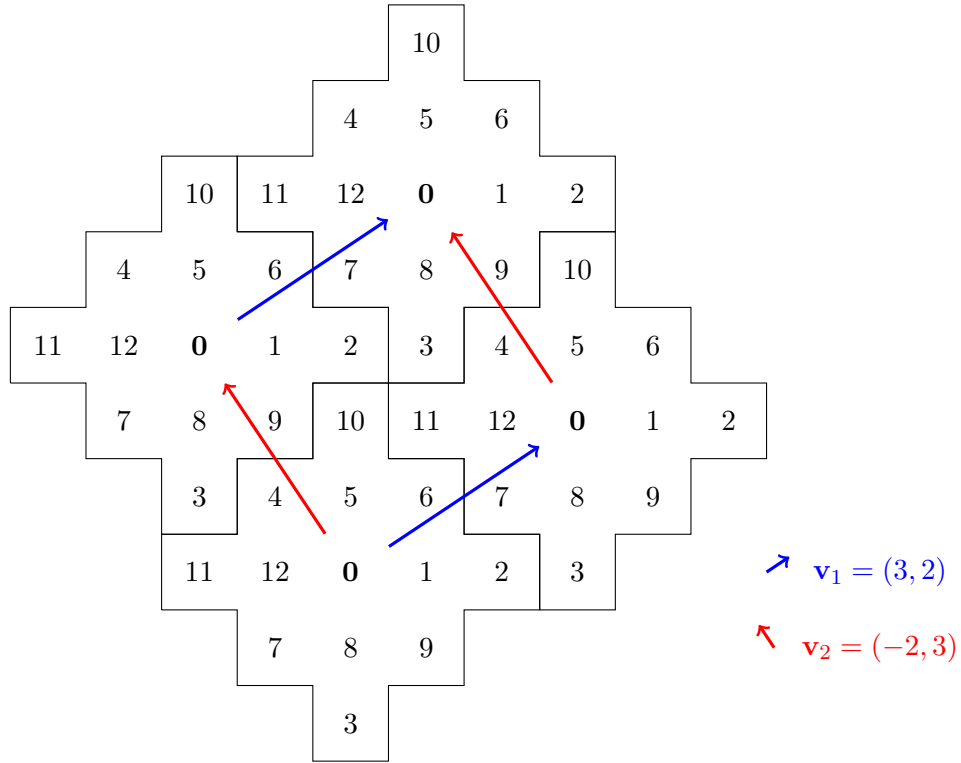
Figure 2.1: Illustration of the relationship between Abelian Cayley graphs and lattice coverings by Lee spheres



In Figure 2.1(a), taking $\mathbf{0}$ as the root vertex, the edges incident to $\mathbf{0}$ are coloured blue, along with its adjacent vertices; the edges incident to these vertices are coloured red, along with their adjacent vertices; with the remaining edges shown in grey. In Figure 2.1(b), centred on $\mathbf{0}$, with horizontal and vertical steps of 1 and 5 to reflect the generators, and with arithmetic modulo 13, the numbers at Manhattan distance 1 from $\mathbf{0}$ are the four connection elements of the graph (1, 5, 8 and 12) and the numbers at distance 2 are the rest; all coloured-coded to match the graph.

As the extremal circulant graph family of degree 4 has order equal to its Abelian Cayley graph upper bound, we would expect to see that the corresponding Lee spheres centred on the corresponding lattice exactly covers \mathbb{Z}^2 with a tiling. Indeed, it proves straightforward to tile the integer plane with these Lee spheres in a way that ensures the horizontal and vertical steps across Lee sphere boundaries. The corresponding lattice vectors are $\mathbf{v}_1 = (3, 2)$ and $\mathbf{v}_2 = (-2, 3)$. Figure 2.2 shows how four neighbouring Lee spheres, centred on neighbouring lattice points, form part of the tiling. The lattice vectors are shown as blue and red arrows.

Figure 2.2: How the Lee spheres centred on lattice points form a tiling for even degree 4, diameter 2



In this case, the order $n = |\mathbb{Z}^f / L_f| = |M| \leq |S_{f,k}|$, where $M = (\mathbf{v}_1, \dots, \mathbf{v}_f)^T$ is the matrix of generating vectors for the lattice L_f . Also, we must have that the Manhattan length of each vector, $\delta(\mathbf{v}_i) \leq 2k + 1$.

Now consider the case where X is an Abelian Cayley graph of odd degree $2f + 1$, so that its connection set includes an involution g_m , and let $\{\mathbf{v}_1, \dots, \mathbf{v}_f\}$ be a set of associated lattice generating vectors. We now define two lattices corresponding to graph X whose union is also a lattice in \mathbb{Z}^f .

Definition 2.3. The *principal lattice* L_f corresponding to an Abelian Cayley graph X of odd degree $2f + 1$ is defined to be the lattice in \mathbb{Z}^f generated by the set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_f\}$ associated with X .

Definition 2.4. The *involutionary lattice* corresponding to an Abelian Cayley graph X of odd degree $2f + 1$ is defined to be the translate of its principal lattice L_f by the *involutionary vector* \mathbf{v}_m , associated with the involutionary graph generator g_m , where $\mathbf{v}_m = \frac{1}{2} \sum \mathbf{v}_i$, thus $L_f + \mathbf{v}_m$. As $2\mathbf{v}_m \in L_f$, the union $L_f \cup (L_f + \mathbf{v}_m)$ is also a lattice.

As $\mathbf{v}_m \in \mathbb{Z}^f$, for each coordinate the sum of the elements of \mathbf{v}_i must be even. The shortest path from an arbitrary vertex to any other vertex either excludes or includes a single occurrence of the involution g_m . The covering of \mathbb{Z}^f is then achieved by the

In this case, the order $n = |\mathbb{Z}^f/L_f| = |M| \leq |S_{f,k}| + |S_{f,k-1}|$, where $M = (\mathbf{v}_1, \dots, \mathbf{v}_f)^T$ is the matrix of generating vectors for the lattice L_f . Also, we must have that the Manhattan distance $\delta(\mathbf{v}_m, \mathbf{v}_i) \leq 2k$.

The extremal circulant graph of degree 5 and diameter 3 has generating set $\{1, 5\}$. It has order 36, just 2 below the Abelian Cayley upper bound of 38. So in this case we would expect to find a covering by the Lee spheres of radius 3 centred on the principal lattice and of radius 2 centred on the involutory lattice. Each pair of spheres overlaps a neighbouring pair translated by one of the lattice vectors in just two common points, so that the covering is almost a tiling. The principal lattice vectors are $\mathbf{v}_1 = (5, -1)$ and $\mathbf{v}_2 = (1, 7)$ and the involutory vector is $\mathbf{v}_m = (3, 3)$. Figure 2.3 shows how four principal Lee spheres are aligned around an involutory sphere. The principal spheres are centred on point $\mathbf{0}$, by definition, and the involutory spheres on point $\mathbf{18}$, being half the order. The two elements 3 and 33 are duplicated in each principal Lee sphere, at the overlap between neighbouring spheres.

Definition 2.6. For an Abelian Cayley graph of dimension f , we define a *lattice generator matrix* to be the $f \times f$ matrix $M = (\mathbf{v}_1, \dots, \mathbf{v}_f)^T$, as defined for Proposition 2.2 for even degree, and for Proposition 2.5 for odd degree. We will often abbreviate *lattice generator matrix* to *LGM*.

Note that the matrix depends on the choice of generating vectors of the lattice, but the lattice is uniquely determined as the kernel of the previously introduced group epimorphism.

2.2 Legacy bounds for extremal undirected Abelian Cayley graphs

We briefly review general upper and lower bounds for the order of extremal undirected Abelian Cayley and circulant graphs of arbitrary degree d and diameter k . For Abelian Cayley graphs, and thus in particular for circulant graphs, an upper bound that is much better than the general Moore bound was established for any given even degree and arbitrary diameter by Wong and Coppersmith in 1974 [51]. An improved upper bound was discovered by Boesch and Wang in 1985, also for given odd degree [2]. These results remain the current best upper bounds.

Unaware of their results, Muga recreated their even-degree result in 1994 [40], Garcia and Peyrat developed an inferior even-degree upper bound in 1997 [15], and Dougherty and Faber reproduced their even and odd-degree results in 2004 [10].

For an undirected Abelian Cayley graph of degree d and diameter k , and corresponding dimension $f = \lfloor d/2 \rfloor$, Boesch and Wang's upper bounds are defined by:

$$Upp_{AbCay}(d, k) = \begin{cases} |S_{f,k}| & \text{for even } d \\ |S_{f,k}| + |S_{f,k-1}| & \text{for odd } d, \end{cases}$$

where, by [48], $|S_{f,k}| = \sum_{i=0}^f 2^i \binom{f}{i} \binom{k}{i}$.

For even or odd degree, this is a polynomial in k of degree f :

$$Upp_{AbCay}(d, k) = \begin{cases} \frac{2^f}{f!} k^f + \frac{2^{f-1}}{(f-1)!} k^{f-1} + \frac{2^{f-2}(f+1)}{3(f-2)!} k^{f-2} + \frac{2^{f-3}f}{3(f-3)!} k^{f-3} + O(k^{f-4}) & \text{for even } d \\ \frac{2^{f+1}}{f!} k^f + \frac{2^{f-1}f(f-1)(f+1)}{3f!} k^{f-2} + O(k^{f-4}) & \text{for odd } d. \end{cases}$$

For even degree, this is equivalent to achieving an exact tiling of \mathbb{Z}^f with Lee spheres of radius k . Such a tiling is possible for dimensions 1 and 2 for any radius, and for any dimension for radius 1. In 1970, Golomb and Welch [17] conjectured that this is not possible for any case with dimension $f \geq 3$ and radius $k \geq 2$. This is the Golomb-Welch Conjecture. It is still open, although various authors have presented proofs of non-existence for 3, 4 and 5 dimensions. The paper by Horak [21] covers all three of these dimensions. However it appears that none of these proofs has established an improved upper bound.

A constructive lower bound for the order of extremal circulant graphs of even degree d and arbitrary diameter k was established by Chen and Jia in 1993 [3]:

$$Low_{circ}(d, k) = \frac{1}{2} \left(\frac{4}{f} \right)^f k^f + \frac{b}{2} \left(\frac{4}{f} \right)^{f-1} k^{f-1} + O(k^{f-2})$$

where $f = d/2$, $k \geq f \geq 3$, and $b \leq 13 - 4f$ with equality if and only if $k \equiv f - 3 \pmod{f}$.

$Low_{circ}(d, k)$ is also a polynomial in k of degree f but with an asymptotically smaller leading coefficient than $Upp_{AbCay}(d, k)$. For even degree d , the ratio of the leading coefficients of $Low_{circ}(d, k)$ and $Upp_{AbCay}(d, k)$, denoted by R_f , is given by

$$R_f = 2^{f-1} \frac{f!}{ff}.$$

We see that $R_1 = R_2 = 1$, and then R_f decreases monotonically to zero with increasing f . The value of R_f measures the asymptotic efficiency of the associated lattice covering. Its reciprocal is the lattice covering density.

More generally, Dougherty and Faber also established a lower bound for the order of extremal Abelian Cayley graphs of even degree d and diameter $k \geq (d-2)/4$, with corresponding dimension $f = d/2$ [10]:

$$Low_{AbCay}(d, k) = \frac{1}{2} \left(\frac{4}{f} \right)^f k^f + \left(\frac{4}{f} \right)^{f-1} k^{f-1} + O(k^{f-2}).$$

This has the same leading coefficients as the Chen and Jia circulant graph lower bound, although their second coefficients are greater.

In contrast to these constructive lower bounds with leading coefficient $(1/2)(4/f)^f$, a theorem in Dougherty and Faber's paper [10] gives a much higher non-constructive lower bound with leading coefficient $\frac{c}{f(\log_e f)^{1+\log_2 e}} \times \frac{2^f}{f!}$, where c is a constant independent of f and k , which is within a factor $1/f^2$ of the upper bound. This significantly sharper lower bound depends on lattice covering theorems by Rogers [46] and Gritzmman [19]. However there is some doubt about the validity of these theorems. This is discussed in Section 13.4.

2.3 Legacy largest-known undirected circulant graph families

Here, we consider only undirected circulant graphs. For dimension 1, degrees 2 and 3, extremal graph families of order $Ext_{circ}(2, k)$ and $Ext_{circ}(3, k)$ are defined trivially. For dimension 2, extremal degree 4 families of order $Ext_{circ}(4, k)$ were discovered by Monakhova in 1979 [36]. Monakhova's paper was only published in Russian and was not easily accessible in other countries. Consequently, several subsequent independent discoveries of the same family were published, beginning with two in 1985, by Boesch and Wang [2] and by Yebra, Fiol, Morillo and Alegre [52]. Extremal degree 5 families of order $Ext_{circ}(5, k)$ were published by Dougherty and Faber in 2004 [10], although they were discovered as early as 1994 according to an earlier preprint [11].

For dimension 3, largest-known circulant graph families of order $LK_{circ}(6, k)$ were discovered for degree 6 by Monakhova in 2003 [37] and independently by Dougherty and Faber, also for degree 7, order $LK_{circ}(7, k)$, in 1994, although not published in a journal until 2004 [10].

The graph orders $Ext_{circ}(2, k)$, $Ext_{circ}(4, k)$ and $LK_{circ}(6, k)$ are all odd for any diameter k . Families of largest-known degree 8 graphs of odd order were discovered

by Monakhova in 2013 and conjectured to be extremal [37]. However, Monakhova had limited her search to odd-order graphs because of a 1994 paper by Muga in which he mistakenly claimed that any extremal circulant graph of even degree and arbitrary diameter has odd order [40]. The argument was flawed, and the smallest counterexample is the extremal graph of degree 8 and diameter 3, which has order 104 (see Table 4.2 in Chapter 4).

Largest-known graph families of degree 8 and 9 were discovered by the author in 2014 [24] and are covered in Chapter 4. For even degree 10 and above, apart from the graphs presented in this thesis, no other circulant graph families have been published in the literature with order above Chen and Jia's lower bound construction $Low_{circ}(d, k)$ [3]. For odd degree 11 and above, there are no other published families.

The upper bounds for Abelian Cayley graphs, $Upp_{AbCay}(d, k)$, are achieved for degrees 2, 3 and 4 by extremal circulant graphs. For degree 2, taking \mathbb{Z}_{2k+1} and generator 1 (so that the connection set $C = \{\pm 1\}$), the resultant graph is the cycle graph on $2k + 1$ vertices which has diameter k , so that $Ext_{circ}(2, k) = 2k + 1$. For degree 3, taking \mathbb{Z}_{4k} and generator 1, connection set $C = \{\pm 1, 2k\}$, the graph is a cycle graph on $4k$ vertices with $2k$ edges added to join opposite pairs of vertices. As $Upp_{AbCay}(3, k) = 4k$, the specified graph is extremal and $Ext_{circ}(3, k) = 4k$.

For degree 4, the Cayley graph of \mathbb{Z}_{2k^2+2k+1} with generating set $\{1, 2k + 1\}$ has diameter k for all k . As $Upp_{AbCay}(4, k) = 2k^2 + 2k + 1$, this proves the graph is extremal and $Ext_{circ}(4, k) = 2k^2 + 2k + 1$. For degree 5, Dougherty and Faber [10] proved that the extremal solution for $k > 1$ is \mathbb{Z}_{4k^2} with generating set $\{1, 2k - 1\}$ (connection set $\{\pm 1, \pm(2k - 1), 2k^2\}$) and order that is 2 less than $Upp_{AbCay}(5, k)$, giving $Ext_{circ}(5, k) = 4k^2$. For degrees 2, 3, 4 and 5, these extremal circulant graphs are also extremal Abelian Cayley graphs.

After degree 5, the situation becomes more difficult. Regarding graphs of three dimensions, Dougherty and Faber [10] verified by computer search that the largest-known families of circulant graphs of degree 6 and 7 are extremal Abelian Cayley graphs for diameter $k \leq 18$ for degree 6, and for diameter $k \leq 10$ for degree 7. For degrees 6 and 7, the formula for the order of the solution, $LK_{circ}(d, k)$, depends on the value of $k \pmod{3}$. Tables 2.1 and 2.2 present these solutions alongside the corresponding expressions for the lower and upper bounds, $Low_{circ}(d, k)$ and $Upp_{AbCay}(d, k)$.

Dougherty and Faber [10] proved the existence of the degree 6 and 7 graphs of order $LK_{circ}(6, k)$ and $LK_{circ}(7, k)$ for all greater values of k , and they remain the largest

Table 2.1: Order of largest-known circulant graph families of degree 6, $LK_{circ}(6, k)$, for arbitrary diameter $k \geq 2$, compared with lower bound $Low_{AbCay}(6, k)$ and upper bound $Upp_{AbCay}(6, k)$

Diameter, k	Family	Order, $LK_{circ}(6, k)$	Lower bound, $Low_{AbCay}(6, k)$
$k \equiv 0 \pmod{3}$	F6:0A	$(32k^3 + 48k^2 + 54k + 27)/27$	$(32k^3 + 48k^2)/27$
$k \equiv 1 \pmod{3}$	F6:1	$(32k^3 + 48k^2 + 78k + 31)/27$	$(32k^3 + 48k^2 + 24k + 4)/27$
$k \equiv 2 \pmod{3}$	F6:2A	$(32k^3 + 48k^2 + 54k + 11)/27$	$(32k^3 + 48k^2 - 16)/27$
Upper bound, $Upp_{AbCay}(6, k) =$			
		$(4k^3 + 6k^2 + 8k + 3)/3 =$	$(36k^3 + 54k^2 + 72k + 27)/27$

Table 2.2: Order of largest-known circulant graph families of degree 7, $LK_{circ}(7, k)$, for arbitrary diameter $k \geq 3$, compared with upper bound $Upp_{AbCay}(7, k)$

Diameter, k	Family	Order, $LK_{circ}(7, k)$
$k \equiv 0 \pmod{3}$	F7:0	$(64k^3 + 108k)/27$
$k \equiv 1 \pmod{3}$	F7:1A	$(64k^3 + 60k - 16)/27$
$k \equiv 2 \pmod{3}$	F7:2A	$(64k^3 + 60k + 16)/27$
Upper bound, $Upp_{AbCay}(7, k) =$		$(8k^3 + 16k)/3 = (72k^3 + 144k)/27$

Note: The lower bound $Low_{AbCay}(d, k)$ is only defined for even degree.

Abelian Cayley graphs of three dimensions so far discovered. For the degree 6 graphs, there is a unique solution up to isomorphism for diameter $k \equiv 1 \pmod{3}$, and for degree 7 there is a unique solution for $k \equiv 0 \pmod{3}$. For other values of k , there are two distinct isomorphism classes of graphs for both degree 6 and 7, where $k \geq 3$ for degree 7.

These polynomials for the order of the graphs are more simply expressed as polynomials in $2a$ in vector notation, for suitable definition of a in terms of k , see Table 2.3.

Table 2.3: Order polynomials for extremal and largest-known circulant graph families up to degree 7 in vector notation ($2a$)

Family	Polynomial in $2a$	where $a =$	Family	Polynomial in $2a$	where $a =$
Dim $f = 1$	$Ext_{circ}(2, k)$			$Ext_{circ}(3, k)$	
F2	$(1 \ 2)/2$	$2k$	F3	$(1 \ 2)$	$2k - 1$
Dim $f = 2$	$Ext_{circ}(4, k)$			$Ext_{circ}(5, k)$	
F4	$(1 \ 2 \ 2)/2$	k	F5	$(1 \ 0 \ 0)$	k
Dim $f = 3$	$LK_{circ}(6, k)$			$LK_{circ}(7, k)$	
F6:0A	$(1 \ 2 \ 3 \ 2)/2$	$2k/3$	F7:2A	$(1 \ 2 \ 3 \ 2)$	$(2k - 1)/3$
F6:1	$(1 \ 0 \ 3 \ 0)/2$	$(2k + 1)/3$	F7:0	$(1 \ 0 \ 3 \ 0)$	$2k/3$
F6:2A	$(1 \ -2 \ 3 \ -2)/2$	$(2k + 2)/3$	F7:1A	$(1 \ -2 \ 3 \ -2)$	$(2k + 1)/3$

Apart from giving simpler coefficients for the dimension 3 polynomials, this notation also reveals symmetries within each degree and between degrees of the same dimension which hint at underlying relationships that will be explored and used to advantage in later chapters.

2.4 Dougherty and Faber's existence proof for circulant families

We summarise the approach taken by Dougherty and Faber in their proof of the existence of the largest-known degree 6 and 7 circulant graph families for arbitrary diameter [10]. The same approach was used for the existence proofs of the degree 8 and 10 circulant graph families, and is applicable for all degrees. It also provided the basis for developing the concept of a lattice generator matrix (LGM), which proves to be a very useful construct for identifying and describing general Abelian Cayley graph families of any degree, as we shall see in later chapters.

As mentioned in Section 2.2 and Proposition 2.1, for any Abelian group A with f generators there is a unique canonical epimorphism from \mathbb{Z}^f onto A with kernel L_f which is an f -dimensional lattice in \mathbb{Z}^f . Then A is isomorphic to \mathbb{Z}^f/L_f , and the Cayley graph of A with given generators is isomorphic to the Cayley graph of \mathbb{Z}^f/L_f with generators being the cosets $\mathbf{e}_i L_f$ for $1 \leq i \leq f$.

In Dougherty and Faber's existence proof for the degree 6 families, they convert the cubic polynomials in the diameter k for each of the three generators of the graph into a vector \mathbf{v}_i in \mathbb{Z}^3 . In order to ensure the vector elements are all integral, a new parameter a is defined, $a = (2k + c)/3$ where c is chosen appropriately for each diameter class. In the general proof for dimension f , $a = (2k + c)/f$. With this substitution, the elements of the vectors all have format $\pm(a \pm b_i)$ for some integers $b_i \geq 0$.

A fourth vector \mathbf{v}_4 is created from a linear combination of the original three so that the eight vectors $\pm\mathbf{v}_i$ give one member of L_3 in each of the octants of \mathbb{Z}^3 . For the dimension f case, the original set of f vectors is extended by linear combination to form a set of 2^{f-1} vectors so that, taken positive and negative, they lie in each orthant of \mathbb{Z}^f .

The proof now requires demonstration that any point \mathbf{x} in \mathbb{Z}^f lies within Manhattan distance k of a lattice point of L_f . By adding or subtracting lattice vectors, \mathbf{x} is translated to a point $\bar{\mathbf{x}}$ close to $\mathbf{0}$ such that each of its coordinates has magnitude at most $a + b_{max}$ where b_{max} is the maximum value of the b_i . If the magnitude of every coordinate is at most $a - b_{max}$, then it is straightforward to prove that \mathbf{x} lies within

distance k of a lattice point. The region that requires specific attention, and the resolution of a large number of special cases that increases rapidly with dimension, is the shell in \mathbb{Z}^f around $\mathbf{0}$ with coordinate magnitudes between $a - b_{max}$ and $a + b_{max}$. Resolving each of these exceptions completes the existence proof. A worked example of all the steps in this proof method can be found in the existence proof of degree 8 circulant graph families, in Section 4.4.

Dougherty and Faber's proof for the degree 7 families is an extension of the even-degree case. The same parameter conversion is applied, from k to a . A principal lattice is established from vectors where the element in one position in turn differs by at most a constant from the value $2a$, while the others have just a constant value. An involutory vector is set equal to half the sum of the three generating vectors, and an involutory lattice is constructed by translating the principal one by the involutory vector, see Proposition 2.5. The proof requires demonstration that any point \mathbf{x} lies within Manhattan distance k of a principal lattice point or within distance $k - 1$ of an involutory lattice point.

In both the even and odd degree case, the original f lattice generating vectors form the rows of an $f \times f$ matrix $M = (\mathbf{v}_1, \dots, \mathbf{v}_f)^T$, the lattice generator matrix. By construction, the determinant of M , $|M| = |\mathbb{Z}^f / L_f|$. Hence, $|M|$ equals the order of the Cayley graph.

2.5 Bounds for extremal directed and mixed Abelian Cayley graphs

For an undirected circulant graph, the dimension f is defined to be the cardinality of a generating set (excluding any involutory generator), so that $f = \lfloor d/2 \rfloor$ where d is the degree.

For directed circulant graphs, the dimension is also defined to be the cardinality of a generating set. However, as the edges are all directed, the generating set contains no inverse pairs or involutions and is identical to the connection set. Thus, the dimension is equal to the directed degree. To clarify the terminology, consider a directed circulant graph with directed degree 1. This means that each vertex has indegree 1, outdegree 1, and therefore total degree 2. When discussing directed or mixed graphs, we will always refer to directed degree rather than total degree.

For mixed circulant graphs of directed degree z and undirected degree d , the dimension f is again the cardinality of a generating set (excluding any involutory generator) given by $f = z + \lfloor d/2 \rfloor$. Clearly, a mixed circulant graph with odd undirected degree must include the involution in its connection set, which means the

order will be even. For non-circulant mixed or undirected Abelian Cayley graphs, there may be multiple involutions, which do not contribute to the dimension. In this case, if the undirected degree $d = r_1 + 2r_2$ where r_1 is the number of involutions and r_2 is the number of self-inverse pairs in the connection set, then the dimension $f = z + r_2$.

Lower and upper bounds for the order of extremal directed circulant graphs of any fixed directed degree and arbitrary diameter were established by Wong and Coppersmith [51] and are presented in Theorem 2.7.

Theorem 2.7. *(Wong and Coppersmith [51]) Let $Ext_{circ}^{dir}(z, k)$ be the order of an extremal directed circulant graph of directed degree z and diameter k . Then for any fixed directed degree $z \geq 2$ and for arbitrary diameter k :*

$$\frac{1}{z^z}k^z + O(k^{z-1}) \leq Ext_{circ}^{dir}(z, k) \leq \binom{z+k}{z} = \frac{1}{z!}k^z + O(k^{z-1}).$$

For dimensions 1 and 2, extremal directed circulant graph families have been discovered (see next section). For dimension 3, the cubic coefficient of the lower and upper bounds for the extremal order are $1/27$ and $1/6$. The lower bound was sharpened by Hsu and Jia [22]:

$$\frac{1}{16}k^3 + \frac{3}{8}k^2 + O(k) \leq Ext_{circ}^{dir}(3, k).$$

Also, the upper bound was sharpened by Fiduccia, Forcade and Zito [14]:

$$Ext_{circ}^{dir}(3, k) \leq \frac{3}{25}(k+3)^3.$$

Beyond dimension 3, Wong and Coppersmith's bounds remain the best.

An upper bound for mixed Abelian Cayley graphs was established by López, Pérez-Rosés and Pujolàs in 2017 [30] and proved using recurrence relations and generating functions. In 2019, Dalfó, Fiol and López provided an elegant derivation of the bound using combinatorial reasoning [8]. Denoting the number of involutions in a generating set by r_1 , the number of pairs of generators and their inverses by r_2 , and the number of additional generators without inverses in the connection set by z , so that the undirected degree is $d = r_1 + 2r_2$ and the directed degree is z , then the upper

bound as a function of the diameter k is

$$Upp_{AbCay}^{mix}(z, r_1, r_2, k) = \sum_{i=0}^k \binom{r_2 + z + i}{i} \binom{r_1 + r_2}{k - i} = \frac{2^{r_1 + r_2}}{(r_2 + z)!} k^{r_2 + z} + O(k^{r_2 + z - 1}).$$

As involutions do not contribute to the dimension, we have the dimension $f = z + r_2$, giving an alternative presentation:

$$Upp_{AbCay}^{mix}(z, r_1, r_2, k) = \frac{2^{r_1 + r_2}}{f!} k^f + O(k^{f-1}).$$

For the undirected case, where $z = 0$, this reduces to the upper bound $Upp_{AbCay}(d, k)$ established by Boesch and Wang [2] (see Section 2.2). For the directed case, where $r_1 = r_2 = 0$, this gives Wong and Coppersmith's upper bound [51] (see above). For general circulant graphs, where the maximum number of involutions is 1, we have a version that depends directly on the undirected degree d :

$$Upp_{circ}^{mix}(z, d, k) = \frac{2^{\lfloor (d+1)/2 \rfloor}}{f!} k^f + O(k^{f-1}).$$

2.6 Legacy largest-known directed circulant graph families

For directed circulant graphs, the dimension is equal to the directed degree. For the trivial case of dimension 1, the extremal directed circulant graph of diameter k is simply the directed cycle graph of order $k + 1$. These form a family valid for all k , with $Ext_{circ}^{dir}(1, k) = k + 1$.

For dimension 2, the quadratic coefficient of the lower and upper bounds for the extremal order are $1/4$ and $1/2$. These values bracket the actual value of $1/3$ for the order of extremal families. They were discovered by Wong and Coppersmith in 1974 [51], although only presented as an improved upper bound. A proof of their existence and extremality was provided in 1994 by Hsu and Jia [22]. The graphs are members of families that are defined for three diameter classes, modulo 3. Their orders are given in the following theorem.

Theorem 2.8. (*Wong and Coppersmith [51], and Hsu and Jia [22]*) *For any diameter $k \geq 2$, the order $Ext_{circ}^{dir}(2, k)$ of the extremal directed circulant graph of directed degree 2 is given by*

$$Ext_{circ}^{dir}(2, k) = \begin{cases} (k^2 + 4k + 3)/3 & \text{for } k \equiv 0 \pmod{3} \\ (k^2 + 4k + 1)/3 & \text{for } k \equiv 1 \pmod{3} \\ (k^2 + 4k + 3)/3 & \text{for } k \equiv 2 \pmod{3}. \end{cases}$$

For diameter class $k \equiv 0 \pmod{3}$ there are two non-isomorphic families, for diameter class 1 there are three, and for class 2 there are two. The existence of multiple isomorphism classes for each diameter class has not been previously published. Formulae for a generating set for each family are shown in Table 2.4.

Table 2.4: Generating sets for extremal directed circulant graph families of directed degree 2 for diameter $k \geq 2$

Family	Diameter class	Generating set g_1 g_2	Odd girth	Girth	Maximal levels	Aut group CI*
H2:0A	0 (mod 3)	1 $k + 3$	$(2k + 3)/3$	$(2k + 3)/3$	$2k/3$	1
H2:0B	0 (mod 6)	1 $(k^2 + k - 3)/3$	$k + 3$	$(2k + 6)/3$	$2k/3$	2
	3 (mod 6)	1 $(k^2 + k - 3)/3$	bipartite	$(2k + 6)/3$	$2k/3$	2
H2:1A	1 (mod 3)	1 $k + 1$	$(2k + 1)/3$	$(2k + 1)/3$	$(2k - 2)/3$	1
H2:1B	1 (mod 6)	1 $k + 4$	bipartite	$(2k + 4)/3$	$(2k - 2)/3$	1
	4 (mod 6)	1 $k + 4$	$k + 1$	$(2k + 4)/3$	$(2k - 2)/3$	1
H2:1C	1 (mod 3)	1 $(k^2 + k - 8)/3$	$(2k + 7)/3$	$(2k + 7)/3$	$(2k - 2)/3$	1
H2:2A	2 (mod 6)	1 $k + 2$	$(5k + 5)/3$	$(2k + 2)/3$	$(2k - 1)/3$	2
	5 (mod 6)	1 $k + 2$	bipartite	$(2k + 2)/3$	$(2k - 1)/3$	2
H2:2B	2 (mod 3)	1 $(k^2 + k)/3$	$(2k + 5)/3$	$(2k + 5)/3$	$(2k - 1)/3$	1

*CI: Cyclic index of the automorphism group

We will return to the topic of directed and mixed circulant graphs of higher dimension in Chapter 15.

THE EXTREMAL ORDER CONJECTURE FOR ABELIAN CAYLEY GRAPHS

In Chapter 3, we present an Extremal Order Conjecture for undirected circulant and Abelian Cayley graphs. The newly discovered graph families described in subsequent chapters are all evaluated against this conjecture. Any graph families with order polynomial as specified for the conjecture are called quasimaximal.

We saw in Section 2.2 that Chen and Jia's lower bound for circulant graphs, $Low_{circ}(d, k)$ [3], and Dougherty and Faber's lower bound for Abelian Cayley graphs, $Low_{AbCay}(d, k)$ [10], share a leading coefficient $(1/2)(4/f)^f$, where d and k are the even degree and the diameter respectively, and $f = d/2$ is the dimension. For dimensions 1 and 2, this coefficient is equal to the leading coefficient, $2^f/f!$, of the Abelian Cayley upper bound $Upp_{AbCay}(d, k)$. But with increasing dimension, their ratio R_f tends to zero exponentially.

It is noteworthy that the largest-known degree 6 families, with order $LK_{circ}(6, k)$, share leading coefficient with the lower bound. Moreover, the graphs in these families have been verified to be extremal as far as checked, up to diameter $k = 18$. This was the basis for an initial conjecture by the author [23] that extremal circulant graph families of any degree have order polynomial with leading coefficient equal to that of the circulant graph lower bound. This conjecture was extended in a later paper [28] to the first two coefficients of the Abelian Cayley lower bound $Low_{AbCay}(d, k)$ and also to cover extremal Abelian Cayley graph families. This leads to the following formal statement of the Extremal Order Conjecture for Abelian Cayley and circulant graphs.

Conjecture 3.1. *Extremal Order Conjecture for Abelian Cayley and circulant graphs.*

Let the order of an extremal Abelian Cayley graph of degree d and diameter k be $Ext_{AbCay}(d, k)$ and similarly $Ext_{circ}(d, k)$ for an extremal circulant graph. Let f be the corresponding dimension $f = \lfloor d/2 \rfloor$ and let K be the diameter class $K = \{k : k \equiv k^ \pmod{f}\}$ for any k^* , $0 \leq k^* < f$.*

Then for any dimension f and diameter class K , $Ext_{AbCay}(d, k)$ and $Ext_{circ}(d, k)$ are polynomials in k of degree f for all $k \in K$ with $k > k_d$ for some threshold value k_d

dependent on d . Moreover

$$Ext_{AbCay}(d, k) = \begin{cases} \frac{1}{2} \left(\frac{4}{f}\right)^f k^f + \left(\frac{4}{f}\right)^{f-1} k^{f-1} + O(k^{f-2}) & \text{for even } d \\ \left(\frac{4}{f}\right)^f k^f + O(k^{f-2}) & \text{for odd } d, \end{cases}$$

$$Ext_{circ}(d, k) = \begin{cases} \frac{1}{2} \left(\frac{4}{f}\right)^f k^f + \left(\frac{4}{f}\right)^{f-1} k^{f-1} + O(k^{f-2}) & \text{for even } d \\ \left(\frac{4}{f}\right)^f k^f + O(k^{f-2}) & \text{for odd } d. \end{cases}$$

Note that the difference $Ext_{AbCay}(d, k) - Ext_{circ}(d, k)$ is at most $O(k^{f-2})$ for any d and k .

The conjecture is true for dimensions 1 and 2, with $k_d = 1$. For dimension 3, its conditions are satisfied by the largest-known circulant graphs, with $k_6 = 1$ and $k_7 = 3$, and also for largest-known Abelian Cayley graphs of dimension 3, which are just the largest-known circulant graphs.

For even degree, both the leading and second terms are identical to those of the Abelian Cayley graph lower bound $Low_{AbCay}(d, k)$. Furthermore, for both even and odd degree, the first two coefficients are the same multiple $R_f = 2^{f-1}(f!/f^f)$ of the corresponding terms of the Abelian Cayley upper bound $Upp_{AbCay}(d, k)$.

Definition 3.2. An Abelian Cayley graph family of arbitrary degree is *quasimaximal* if the first two coefficients of its order polynomial in the diameter are equal to those of the Extremal Order Conjecture. A family with a lower first coefficient, or an equal first coefficient and lower second coefficient is said to be *subquasimaximal*.

If Conjecture 3.1 is correct, then this implies the following formulae for the order of extremal circulant graphs of degrees 8 to 11 above some threshold diameter:

$$Ext_{circ}(d, k) = \begin{cases} (k^4 + 2k^3)/2 + O(k^2) & \text{for } d = 8 \\ k^4 + 3k^2 + O(k^2) & \text{for } d = 9 \\ (512k^5 + 1280k^4)/3125 + O(k^3) & \text{for } d = 10 \\ 1024k^5/3125 + O(k^3) & \text{for } d = 11. \end{cases}$$

As we shall see in the next chapter, largest-known circulant graph families have been discovered with precisely these coefficients. Indeed, quasimaximal circulant graph families have been discovered for degrees up to 20 and beyond.

A much higher lower bound, depending on a theorem by Rogers [46], was mentioned in Section 2.2 and conflicts with this conjecture. If the Extremal Order Conjecture 3.1 is correct, then Rogers' theorem is invalid. An alternative position might be that Rogers' theorem is valid asymptotically and the Extremal Order Conjecture is valid only up to some threshold dimension. This question is addressed in Section 13.4.

LARGEST-KNOWN CIRCULANT GRAPH

FAMILIES OF DEGREES 6 TO 11

In Chapter 4, newly discovered largest-known circulant graph families up to degree 11 are described, along with a proof of the existence of the degree 8 families and lattice generating vectors for the degree 10 proof. The degree 8 and 9 families and the degree 8 existence proof were previously the subject of the author's MSc dissertation. As stated in Definition 1.4, a graph family is an infinite set of graphs of given degree d and dimension $f = \lfloor d/2 \rfloor$, defined for each diameter k of a diameter class, with order and generating set specified by polynomials in k of maximum degree f .

4.1 Dimension 3, degrees 6 and 7

We already noted in Section 2.3 that largest-known circulant graph families of degree 6 and 7 for the three diameter classes 0, 1 and 2 were discovered by Monakhova [37] and Dougherty and Faber [10]. However, for certain diameter classes there is a second, non-isomorphic family with the same order that has been discovered by the author and has not previously been documented. These families exist for degree 6, diameter classes 0 and 2, and for degree 7, diameter classes 1 and 2. For reference, the legacy graph families are labelled with suffix A and these new discoveries with suffix B. The new families are presented below, in Table 4.1. Note the similarity of the polynomials in each row between degrees 6 and 7. This is structural and is discussed in Chapter 5.

Table 4.1: Order and generating set polynomials for newly discovered largest-known circulant graph families of degrees 6 and 7, diameter $k \pmod{3}$

Degree 6 family	Order and generators	Polynomial in $2a$	Degree 7 family	Order and generators	Polynomial in $2a$
F6:0B $k \equiv 0$ $a = 2k/3$	Order g_1 g_2 g_3	$(1\ 2\ 3\ -2)/2$ $(0\ 0\ 1\ -1)$ $(0\ 0\ 1\ 1)$ $(0\ 1\ 0\ 1)$	F7:2B $k \equiv 2$ $a = (2k - 1)/3$	Order g_1 g_2 g_3	$(1\ 2\ 3\ 2)$ $(0\ 0\ 1\ -1)$ $(0\ 0\ 1\ 1)$ $(0\ 1\ 0\ 1)$
F6:2B $k \equiv 2$ $a = (2k + 2)/3$	Order g_1 g_2 g_3	$(1\ -2\ 3\ -2)/2$ $(0\ 0\ 1\ 1)$ $(0\ 0\ 1\ -1)$ $(0\ 1\ 0\ 1)$	F7:1B $k \equiv 1$ $a = (2k + 1)/3$	Order g_1 g_2 g_3	$(1\ -2\ 3\ -2)$ $(0\ 0\ 1\ 1)$ $(0\ 0\ 1\ -1)$ $(0\ 1\ 0\ 1)$

Formulae for the order and a generating set for these families and the other largest-known families of degrees 6 and 7 are presented in Appendix A.2. Properties of the graphs up to diameter 16 are given in Appendix D, Tables D.5 and D.6.

4.2 Dimension 4, degrees 8 and 9

Graphs of dimension 4 have degree 8 or 9. This work was originally undertaken by the author for an MSc dissertation and was published in 2014 [23]. As with Dougherty and Faber’s approach for dimension 3, an exhaustive computer search was conducted for potential solutions using all feasible generating sets within relevant ranges. For small diameter, this process worked well and enabled the discovery of families of graphs of degree 8 that are just larger than Monakhova’s largest-known odd-order graphs, [37]. The process was similarly successful for degree 9, although with no lower bound for comparison.

The order of graphs on generator sets of dimension 4 increases with diameter much more quickly than for dimension 3, as well as the number of possible permutations for each order. This means that the calculations to prove the extremality of a candidate graph by continuing the search up to the relevant upper bound, $Upp_{AbCay}(d, k)$, quickly exceed the available computing power. Therefore, the discovered candidate families of dimension 4 graphs have only been verified to be extremal for a rather limited range of diameters, $k \leq 7$ for degree 8 and $k \leq 6$ for degree 9. The results for degree 8, up to diameter 16, are shown in Table 4.2. See also Appendix D, Table D.7.

Table 4.2: Largest-known circulant graphs of degree 8

Diameter k	Order $LK_{circ}(8, k)$	Family (F) or graph (G)	Generating set*	Upper bound $Upp_{AbCay}(8, k)$	Status
2	35	G8:2A G8:2B	1, 6, 7, 10 1, 7, 11, 16	41	Extremal
3	104	F8:1	1, 16, 20, 27	129	Extremal
4	248	F8:0	1, 61, 72, 76	321	Extremal
5	528	F8:1	1, 89, 156, 162	681	Extremal
6	984	F8:0	1, 163, 348, 354	1289	Extremal
7	1712	F8:1	1, 215, 608, 616	2241	Extremal
8	2768	F8:0	1, 345, 1072, 1080	3649	Largest known
9	4280	F8:1	1, 429, 1660, 1670	5641	Largest known
10	6320	F8:0	1, 631, 2580, 2590	8361	Largest known
11	9048	F8:1	1, 755, 3696, 3708	11969	Largest known
12	12552	F8:0	1, 1045, 5304, 5316	16641	Largest known
13	17024	F8:1	1, 1217, 7196, 7210	22569	Largest known
14	22568	F8:0	1, 1611, 9772, 9786	29961	Largest known
15	29408	F8:1	1, 1839, 12736, 12752	39041	Largest known
16	37664	F8:0	1, 2353, 16608, 16624	50049	Largest known

* for each isomorphism class of graphs just one of the generating sets is listed

The diameter 2 solution was found by McKay [33] who also discovered the diameter 2 solutions up to degree 16. The solutions for diameters 3 to 5 were found by Fera-Purón, Ryan and Pérez-Rosés [13] and independently by the author. The graphs of diameter 6 and above were discovered by the author.

For degree 8, the following quartic polynomials in k determine the order of these solutions for diameter $k \geq 3$:

$$LK_{circ}(8, k) = \begin{cases} (k^4 + 2k^3 + 6k^2 + 4k)/2 & \text{for } k \equiv 0 \pmod{2} \\ (k^4 + 2k^3 + 6k^2 + 6k + 1)/2 & \text{for } k \equiv 1 \pmod{2}. \end{cases}$$

Over the range of diameters checked, there is just one unique graph up to isomorphism for each $k \geq 3$. The leading coefficient of $1/2$ equals the lower bound value in the formula for $Low_{circ}(8, k)$ and is below the upper bound value of $2/3$ in $Upp_{AbCay}(8, k)$. The first two coefficients are consistent with the Extremal Order Conjecture, 3.1. Formulae for generating sets are shown in Table 4.3.

Table 4.3: Order and generating sets of largest-known circulant graph families of degree 8 for diameter $k \geq 3$

		Family F8:0 $k \equiv 0 \pmod{2}$	Family F8:1 $k \equiv 1 \pmod{2}$
Order, $LK_{circ}(8, k)$		$(k^4 + 2k^3 + 6k^2 + 4k)/2$	$(k^4 + 2k^3 + 6k^2 + 6k + 1)/2$
Generating set	g_1	1	1
	g_2	$(k^3 + 2k^2 + 6k + 2)/2$	$(k^3 + k^2 + 5k + 3)/2$
	g_3	$(k^4 + 4k^2 - 8k)/4$	$(k^4 + 2k^2 - 8k - 11)/4$
	g_4	$(k^4 + 4k^2 - 4k)/4$	$(k^4 + 2k^2 - 4k - 7)/4$

For $k = 2$, the formula gives a graph of order 32 whereas the extremal order is 35, with two non-isomorphic solutions. For $3 \leq k \leq 7$, the resulting graphs have been verified to be extremal by exhaustive computer search up to the upper bound $Upp_{AbCay}(8, k)$. The existence of these graphs for all k is proved in Section 4.4. They are the largest-known degree 8 circulant graph families so far discovered for any $k \geq 3$ and are conjectured to be extremal.

The results for degree 9, up to diameter 16, are shown in Table 4.4. See also Appendix D, Table D.8. The diameter 2 solution was found by McKay [33]. The solutions for diameters 3 and 4 were found by Fera-Purón, Ryan and Pérez-Rosés [13], and diameter 4 independently by the author. The graph of diameter 5 and above were discovered by the author.

Table 4.4: Largest-known circulant graphs of degree 9

Diameter k	Order $LK_{circ}(9, k)$	Family (F) or graph (G)	Generating set*	Upper bound $Upp_{AbCay}(9, k)$	Status
2	42	G9:2A	1, 5, 14, 17	50	Extremal
		G9:2B	2, 7, 8, 10		
3	130	G9:3A	1, 8, 14, 47	170	Extremal
		G9:3B	1, 8, 20, 35		
		G9:3C	1, 26, 49, 61		
		G9:3D	2, 8, 13, 32		
4	320	G9:4	1, 15, 25, 83	450	Extremal
5	700	F9:1a	1, 5, 197, 223	1002	Extremal
		F9:1b	1, 45, 225, 231		
6	1416	F9:0	1, 7, 575, 611	1970	Extremal
7	2548	F9:1a	1, 7, 521, 571	3530	Largest known
		F9:1b	1, 581, 1021, 1029		
8	4304	F9:0	1, 9, 1855, 1919	5890	Largest known
9	6804	F9:1a	1, 9, 1849, 1931	9290	Largest known
		F9:1b	1, 1305, 1855, 1863		
10	10320	F9:0	1, 11, 4599, 4699	14002	Largest known
11	15004	F9:1a	1, 11, 3349, 3471	20330	Largest known
		F9:1b	1, 4851, 6655, 6667		
12	21192	F9:0	1, 13, 9647, 9791	28610	Largest known
13	29068	F9:1a	1, 13, 7741, 7911	39210	Largest known
		F9:1b	1, 5083, 7929, 7943		
14	39032	F9:0	1, 15, 18031, 18227	52530	Largest known
15	51300	F9:1a	1, 15, 11857, 12083	69002	Largest known
		F9:1b	1, 5835, 15075, 15089		
16	66336	F9:0	1, 17, 30975, 31231	89090	Largest known

* for each isomorphism class of graphs just one of the generating sets is listed

For degree 9, the following quartic polynomials in k determine the order of the largest-known solutions for diameter $k \geq 5$:

$$LK_{circ}(9, k) = \begin{cases} k^4 + 3k^2 + 2k & \text{for } k \equiv 0 \pmod{2} \\ k^4 + 3k^2 & \text{for } k \equiv 1 \pmod{2}. \end{cases}$$

This may be compared with the upper bound $Upp_{AbCay}(9, k) = (4k^4 + 20k^2 + 6)/3$.

Here also, the first two coefficients are consistent with the Extremal Order Conjecture, 3.1.

Over the range of diameters $k \geq 5$ checked, there is a unique family for each even diameter (F9:0) and two for each odd diameter (F9:1a and F9:1b). For their formulae, see Table 4.5.

Formulae for order and a generating set for these largest-known circulant graph families of degrees 8 and 9 are presented in vector notation as polynomials in $2a$ in

Table 4.5: Order and generating sets of largest-known circulant graph families of degree 9 for diameter $k \geq 5$

		Family F9:0 $k \equiv 0 \pmod{2}$	Family F9:1a $k \equiv 1 \pmod{2}$	Family F9:1b $k \equiv 1 \pmod{2}$
Order, $LK_{circ}(9, k)$		$k^4 + 3k^2 + 2k$	$k^4 + 3k^2$	$k^4 + 3k^2$
Generating set	g_1	1	1	k^2
	g_2	$k + 1$	$k^3 + 2k$	$k^3 + 2k - 2$
	g_3	$(k^4 - k^3 + 2k^2 - 2)/2$	$k^3 + 3k + 1$	$k^3 + 2k$
	g_4	$(k^4 - k^3 + 4k^2 - 2)/2$	$k^3 + k^2 + 3k + 2$	$k^3 + 4k - 2$

Appendix A.3. Properties of the individual graphs up to diameter 16 are given in Appendix D, Tables D.7 and D.8.

4.3 Dimension 5, degrees 10 and 11

The author's work on graphs of dimension 5 (that is, of degree 10 or 11) was published in 2018 [26]. The process that was followed to discover the largest-known degree 10 and 11 graphs and the quintic polynomials in the diameter that define their orders and generating sets was an extension of the methods used by Dougherty and Faber for the degree 6 and 7 families and by the author for degrees 8 and 9. It involved a combination of four methods: assumption of the validity of the Extremal Order Conjecture (Conjecture 3.1), analysis of the largest-known families of smaller degree to discover common features that may extrapolate, computer searches that needed to be increasingly focused as the diameter increased, and a measure of inspiration and pattern recognition. These are discussed in more detail below.

If true, the Extremal Order Conjecture implies that the extremal graphs of degree 10 would have order defined by quintic polynomials in the diameter, one for each diameter class $k \pmod{5}$, and also specifies their common first two coefficients, reducing the degrees of freedom accordingly. As with the approach for degrees 6 to 9, for small diameter the extremality of the graphs was confirmed by conducting a computer search using feasible generating sets for graphs of every order up to the upper bound. However, the number of possible permutations of elements for generating sets of dimension 5 increases rapidly with diameter, quickly exceeding available computing power. For degree 10, the graphs could only be verified extremal up to diameter 5, which provided a maximum of only one graph as the basis for each of the five presumed families.

From an analysis of the largest-known families of smaller degree, common factors were discovered that were tentatively assumed to remain valid, greatly reducing the search

space for the computer runs. For example, as mentioned earlier, every graph in a largest-known circulant graph family up to degree 9 has a primitive generating set, so the computer searches were set to fix one of the generators at 1, eliminating a degree of freedom. Although all Abelian Cayley graphs have girth 3 or 4 by definition, all the largest-known families have an odd girth that is maximum $(2k + 1)$, and so the computer searches were restricted to maximum odd girth, which reduced the run-time significantly. Also, for each degree $d \leq 9$ the order polynomials for the families for diameter $k \equiv 0 \pmod f$ have value 0 or 1 for $k = 0$ depending on the parity of the order of the family, again reducing the degree of freedom. It was also observed that generating sets of largest-known families often include pairs of generators differing by a small value that increases linearly with diameter. Where this was found to occur for a single graph in a family, the subsequent search for other graphs in the prospective family was restricted to include such pairs, further reducing the degree of freedom.

Even utilising these and other similar techniques, the process remained complex and time-consuming. Each newly discovered graph became a potential member of its family, restricting the freedom of the corresponding order polynomial and further sharpening the search for the next graph in the family. Any subsequent search failure would require backtracking to eliminate a candidate graph and initiate a search for

Table 4.6: Largest-known degree 10 circulant graphs, up to diameter 16

Diameter k	Order $LK_{circ}(10, k)$	Family (F, O) or graph (G)	Generating set*	Status
2	51	G10:2	1, 2, 10, 16, 23	Extremal
3	177	G10:3	1, 12, 19, 27, 87	Extremal
4	457	F10:4	1, 20, 130, 147, 191	Extremal
5	1099	F10:0	1, 53, 207, 272, 536	Extremal
6	2380	F10:1	1, 555, 860, 951, 970	Largest known
	2329	O10:1 (odd)	1, 75, 390, 453, 764	Largest-known odd
7	4551	F10:2	1, 739, 1178, 1295, 1301	Largest known
8	8288	F10:3	1, 987, 2367, 2534, 3528	Largest known
	8183	O10:3A (odd)	1, 286, 294, 1707, 3758	Largest-known odd
		O10:3B (odd)	1, 112, 120, 953, 1504	
9	14099	F10:4	1, 247, 1766, 1983, 3494	Largest known
10	22805	F10:0	1, 313, 2495, 2846, 5662	Largest known
11	35568	F10:1	1, 4347, 7470, 7903, 11808	Largest known
	35243	O10:1 (odd)	1, 387, 3528, 3877, 7010	Largest-known odd
12	53025	F10:2	1, 5251, 19281, 19291, 19806	Largest known
13	77572	F10:3	1, 6347, 14103, 14740, 21098	Largest known
	77077	O10:3A (odd)	1, 1594, 21165, 36774, 36784	Largest-known odd
		O10:3B (odd)	1, 4344, 29303, 38093, 38103	
14	110045	F10:4	1, 827, 9176, 9935, 18272	Largest known
15	152671	F10:0	1, 973, 11663, 12716, 25364	Largest known
16	208052	F10:1	1, 17147, 30784, 32007, 47918	Largest known
	207037	O10:1 (odd)	1, 1131, 14794, 15845, 29496	Largest-known odd

* for each graph family just one of the generating sets is listed

the next candidate. The challenge eventually yielded to the effort, however, as each graph family in turn was completed, for degree 10 and similarly for degree 11.

The largest-known degree 10 circulant graphs up to diameter 16 are shown in Table 4.6. For diameters 2 to 4, these results are not new, as documented in Combinatorics Wiki [7]. The graphs of diameter 5 and above were discovered by the author. For each diameter class, there is a single family of largest-known graphs, each with at least one primitive generating set. Note that for diameter $k \equiv 1$ or $3 \pmod{5}$, when $k \geq 6$ the largest-known graphs have even order. For completeness, the largest-known degree 10 circulant graphs of odd order and diameter $k \equiv 1$ and $k \equiv 3 \pmod{5}$ are also included in the table. See also Appendix D, Table D.9.

The following quintic polynomials in k determine the order of these largest-known circulant graph families for arbitrary diameter $k \geq 4$. $LK_{circ}(10, k) =$

$$\begin{cases} (512k^5 + 1280k^4 + 6400k^3 + 8000k^2 + 6250k + 3125)/3125 & \text{for } k \equiv 0 \pmod{5} \\ (512k^5 + 1280k^4 + 6560k^3 + 9520k^2 + 6100k + 1028)/3125 & \text{for } k \equiv 1 \pmod{5} \\ (512k^5 + 1280k^4 + 6080k^3 + 7840k^2 + 10010k + 3741)/3125 & \text{for } k \equiv 2 \pmod{5} \\ (512k^5 + 1280k^4 + 6560k^3 + 7600k^2 + 4180k + 1344)/3125 & \text{for } k \equiv 3 \pmod{5} \\ (512k^5 + 1280k^4 + 6400k^3 + 8640k^2 + 6890k + 757)/3125 & \text{for } k \equiv 4 \pmod{5}. \end{cases}$$

The largest-known circulant graph families of odd order for diameters $k \equiv 1$ and $k \equiv 3 \pmod{5}$ have order given by the following quintics:

$$\begin{cases} (512k^5 + 1280k^4 + 5760k^3 + 9920k^2 + 6450k - 2047)/3125 & \text{for } k \equiv 1 \pmod{5} \\ (512k^5 + 1280k^4 + 5760k^3 + 8800k^2 + 4830k + 819)/3125 & \text{for } k \equiv 3 \pmod{5}. \end{cases}$$

The graphs of all these families have orders that share common leading and second coefficients, $512/3125$ and $1280/3125$, consistent with the Extremal Order Conjecture, 3.1.

These formulae define the order $LK_{circ}(10, k)$ of the largest-known circulant graphs of degree 10 for any diameter $k \geq 4$. For diameter $k \leq 3$, the graphs with order defined by these formulae are not extremal. For $k = 2$, the formula gives a graph of order 45 whereas the extremal order is 51; for $k = 3$, the formula gives 156 instead of 177.

Proof of the existence of these graph families for all $k \geq 4$ is addressed in Section 4.5. They are the largest degree 10 circulant graph families discovered and are conjectured to be extremal for $k \geq 4$.

The process that was followed to discover the largest-known degree 11 graphs and the quintic polynomials in the diameter that define their orders and generating sets was the same as for degree 10. The largest-known degree 11 circulant graphs up to diameter 16 are shown in Table 4.7. See also Appendix D, Table D.10. For diameter $k = 2$, there are five distinct isomorphism classes, one of which does not have a primitive generating set. For $k = 3$, there are two classes, including one with no primitive generating set, and just one for diameter 4. For diameters 2 to 4, these results are not new, as documented in Combinatorics Wiki [7]. The graphs of diameter 5 and above were discovered by the author. For higher diameters, there are single graph families for diameter classes 0, 2, 3, and 4, with a primitive generating set, and two distinct families for diameter class 1, labelled F11:1a and F11:1b.

Table 4.7: Largest-known degree 11 circulant graphs, up to diameter 16

Diameter k	Order $LK_{circ}(11, k)$	Family (F) or graph (G)	Generating set*	Status
2	56	G11:2A	1, 2, 10, 15, 22	Extremal
		G11:2B	1, 4, 6, 15, 24	
		G11:2C	1, 6, 10, 15, 18	
		G11:2D	1, 9, 14, 21, 25	
		G11:2E	2, 6, 7, 18, 21	
3	210	G11:3A	1, 49, 59, 84, 89	Extremal
		G11:3B	2, 32, 63, 92, 98	
4	576	G11:4	1, 9, 75, 155, 179	Largest known
5	1428	F11:0	1, 169, 285, 289, 387	Largest known
6	3200	F11:1a	1, 101, 925, 1031, 1429	Largest known
		F11:1b	1, 265, 851, 1111, 1321	
7	6652	F11:2	1, 107, 647, 2235, 2769	Largest known
8	12416	F11:3	1, 145, 863, 4163, 5177	Largest known
9	21572	F11:4	1, 189, 1517, 8113, 9435	Largest known
10	35880	F11:0	1, 2209, 5127, 5135, 12537	Largest known
11	56700	F11:1a	1, 1053, 1061, 10603, 17965	Largest known
		F11:1b	1, 4113, 4121, 13301, 23723	
12	87248	F11:2	1, 479, 4799, 34947, 39257	Largest known
13	128852	F11:3	1, 581, 5799, 51599, 57989	Largest known
14	184424	F11:4	1, 693, 8325, 76901, 84523	Largest known
15	259260	F11:0	1, 10729, 39875, 39887, 90637	Largest known
16	355576	F11:1a	1, 22307, 131327, 136371, 153621	Largest known
		F11:1b	1, 8579, 75569, 75583, 111513	

* excludes the involution; for each graph family just one of the generating sets is listed

The following quintic polynomials in k determine the order of these graph families for any arbitrary diameter $k \geq 5$. $LK_{circ}(11, k) =$

$$\begin{cases} (1024k^5 + 9600k^3 + 12500k)/3125 & \text{for } k \equiv 0 \pmod{5} \\ (1024k^5 + 8960k^3 + 2880k^2 - 260k - 104)/3125 & \text{for } k \equiv 1 \pmod{5} \\ (1024k^5 + 10240k^3 + 640k^2 + 5140k - 2528)/3125 & \text{for } k \equiv 2 \pmod{5} \\ (1024k^5 + 10240k^3 - 640k^2 + 5140k + 2528)/3125 & \text{for } k \equiv 3 \pmod{5} \\ (1024k^5 + 8960k^3 + 5120k^2 + 740k - 6896)/3125 & \text{for } k \equiv 4 \pmod{5}. \end{cases}$$

Graphs with these orders have been constructed for all diameters up to $k = 100$. They are the largest degree 11 circulant graphs discovered for any $k \geq 5$ and are conjectured to be extremal. Above diameter $k = 100$, the existence of graphs with these constructions is confirmed by the Existence Proof Theorem 6.16 in Chapter 6. They are conjectured to be extremal for all higher diameters. The graphs of all these families have orders that share common leading and second coefficients, $1024/3125$ and 0, consistent with the Extremal Order Conjecture, 3.1.

For diameter $k \leq 4$, the graphs with order defined by these formulae are not extremal. For $k = 2, 3, 4$ respectively, the formulae give graphs of order 40, 172 and 544, whereas the extremal orders are 56, 210 and 576.

The polynomials for the order of these degree 10 and 11 graph families are more simply expressed as polynomials in $2a$ in vector notation, with a suitably defined in terms of k , see Table 4.8. This format also reveals inherent relationships between diameter classes within a degree and between the degrees that are not apparent from the polynomials in k . These will be discussed in the next chapter.

Table 4.8: Order of largest-known circulant graph families of degrees 10 and 11

Deg 10 family	Order polynomial in $2a$	where $a =$	Deg 11 family	Order polynomial in $2a$	where $a =$
F10:0	(1 2 8 8 5 2)/2	$2k/5$	F11:3	(1 2 8 8 5 2)	$(2k - 1)/5$
F10:1	(1 -4 13 -20 14 -4)/2	$(2k + 3)/5$			
O10:1	(1 -4 12 -16 9 -4)/2		F11:4	(1 -4 12 -16 9 -4)	$(2k + 2)/5$
F10:2	(1 0 6 0 5 0)/2	$(2k + 1)/5$	F11:0	(1 0 6 0 5 0)	$2k/5$
F10:3	(1 4 13 20 14 4)/2	$(2k - 1)/5$			
O10:3	(1 4 12 20 15 4)/2		F11:1	(1 4 12 20 15 4)	$(2k - 2)/5$
F10:4	(1 -2 8 -8 5 -2)/2	$(2k + 2)/5$	F11:2	(1 -2 8 -8 5 -2)	$(2k + 1)/5$

In order to simplify the presentation and reveal inherent relationships, all generating sets will be shown in vector notation. Generating sets for the five largest-known degree 10 circulant graph families are presented in Table 4.9

Table 4.9: Generating sets for largest-known degree 10 circulant graph families

Family	Polynomial in $2a$	where $a =$	Family	Polynomial in $2a$	where $a =$
F10:0	g_1 (0 0 1 3 2 0)/2	$2k/5$	F10:4	g_1 (0 0 1 -3 2 0)/2	$(2k + 2)/5$
	g_2 (0 0 2 0 0 0)/2			g_2 (0 0 2 0 0 0)/2	
	g_3 (0 0 2 3 3 2)/2			g_3 (0 0 2 -3 3 -2)/2	
	g_4 (0 1 2 4 5 2)/2			g_4 (0 1 -2 4 -5 2)/2	
	g_5 (0 1 1 2 0 0)/2			g_5 (0 1 -1 2 0 0)/2	
F10:1	g_1 (0 0 1 -4 0 2)/2	$(2k + 3)/5$	F10:3	g_1 (0 0 1 4 0 -2)/2	$(2k - 1)/5$
	g_2 (0 1 -2 6 -10 6)/2			g_2 (0 1 2 6 10 6)/2	
	g_3 (0 1 -6 14 -23 14)/2			g_3 (0 1 6 14 23 14)/2	
	g_4 (0 2 -6 15 -13 2)/2			g_4 (0 2 6 15 13 2)/2	
	g_5 (0 2 -7 25 -38 18)/2			g_5 (0 2 7 25 38 18)/2	
F10:2	g_1 (0 0 0 2 0 2)/2	$(2k + 1)/5$			
	g_2 (0 0 1 0 3 -2)/2				
	g_3 (0 0 1 0 3 2)/2				
	g_4 (0 1 0 5 -2 2)/2				
	g_5 (0 1 0 5 2 2)/2				

These generating sets are not primitive (the sets do not include 1). If the generators in such a set are multiplied by any factor up to the order of the graph, then the resultant set will be the generating set for an isomorphic graph, as long as the factor is coprime with the order. At most one valid factor will take any generator to the value of 1. Therefore, the maximum number of primitive generating sets for an isomorphic graph is equal to the dimension of the graph. Some graph families have generating sets that have the maximum number of primitive equivalents for all diameters of their diameter class. At the other extreme, some have none or only one, while others have a regularly varying number as the diameter increases.

Although all five largest-known degree 10 circulant graph families have primitive generating sets for all diameters, some do not have a single set of formulae valid for all diameters within their diameter class. Therefore, it is more efficient in presentation to accept imprimitive generating sets. It is a relatively simple exercise to calculate the primitive generating sets from any given imprimitive one. For the degree 10 case, primitive generating sets are presented in [26] with a more comprehensive listing in [28].

Generating sets for the five largest-known degree 11 circulant graph families are presented in Table 4.10

Table 4.10: Generating sets for largest-known degree 11 circulant graph families

Family	Polynomial in $2a$	where $a =$	Family	Polynomial in $2a$	where $a =$
F11:0	g_1 (0 1 0 2 0 1) g_2 (0 1 0 4 -4 -1) g_3 (0 1 0 4 4 -1) g_4 (0 1 -2 6 -6 1) g_5 (0 1 2 6 6 1)	$2k/5$	F11:1a	g_1 (0 1 4 12 16 7) g_2 (0 1 4 10 14 3) g_3 (0 1 2 6 4 3) g_4 (0 1 4 12 22 9) g_5 (0 1 2 8 4 1)	$(2k-2)/5$
F11:4	g_1 (0 1 0 4 0 1) g_2 (0 1 -4 8 -4 -3) g_3 (0 1 -4 8 -4 5) g_4 (0 1 -4 8 -12 5) g_5 (0 1 -4 16 -12 5)	$(2k+2)/5$	F11:1b	g_1 (0 1 6 18 26 13) g_2 (0 1 4 8 12 5) g_3 (0 1 2 10 16 9) g_4 (0 1 2 4 4 3) g_5 (0 1 2 4 -2 -3)	$(2k-2)/5$
F11:2	g_1 (0 1 0 -2 2 -1) g_2 (0 1 -2 2 -2 1) g_3 (0 1 2 -2 2 -1) g_4 (0 1 -4 4 -2 1) g_5 (0 1 -4 6 -4 1)	$(2k+1)/5$	F11:3	g_1 (0 1 0 -2 -2 -1) g_2 (0 1 2 2 2 1) g_3 (0 1 -2 -2 -2 -1) g_4 (0 1 4 4 2 1) g_5 (0 1 4 6 4 1)	$(2k-1)/5$

Formulae for order and a generating set for these largest-known circulant graph families of degrees 10 and 11 are presented as polynomials in $2a$ in Appendix A.4. Properties of the individual graphs up to diameter 16 are given in Appendix D, Tables D.9 and D.10.

4.4 Existence proof of the degree 8 families

In this section, we prove the existence of the largest-known degree 8 circulant graph families of order $LK_{\text{circ}}(8, k)$ for all diameters k . The method of proof closely follows the approach taken by Dougherty and Faber in their proof of the existence of the largest-known degree 6 graph families of order $LK_{\text{circ}}(6, k)$ [10]. For both diameter classes $k \equiv 0$ and $1 \pmod{2}$ all stages of the proof are presented. However, resolution of the residual boundary exceptions is only included for the first orthant of the solution space, as an example. This abridged version of the proof was published in 2014 [23]. The full set of boundary exception resolutions for all orthants can be found on arXiv [27].

Theorem 4.1. *For all $k \geq 2$, there is an undirected Cayley graph on four generators of a cyclic group which has diameter k and order $LK_{\text{circ}}(8, k)$, where*

$$LK_{\text{circ}}(8, k) = \begin{cases} (k^4 + 2k^3 + 6k^2 + 4k)/2 & \text{if } k \equiv 0 \pmod{2} \\ (k^4 + 2k^3 + 6k^2 + 6k + 1)/2 & \text{if } k \equiv 1 \pmod{2}. \end{cases}$$

Moreover, for $k \equiv 0 \pmod{2}$, a generating set is

$$\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k)/4\},$$

and for $k \equiv 1 \pmod{2}$,

$$\{1, (k^3 + k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}.$$

Proof. We will show the existence of four-dimensional lattices $L_k \subseteq \mathbb{Z}^4$ such that \mathbb{Z}^4/L_k is cyclic, $S_{f,k} + L_k = \mathbb{Z}^4$, where $S_{f,k}$ is the set of points in \mathbb{Z}^4 at a distance of at most k from the origin under the Manhattan norm, and $|\mathbb{Z}^4 : L_k| = LK_{circ}(8, k)$ as specified in the theorem. Then, by Proposition 2.1, the resultant Cayley graph has diameter at most k .

$$\text{Let } a = \begin{cases} k/2 & \text{for } k \equiv 0 \pmod{2} \\ (k+1)/2 & \text{for } k \equiv 1 \pmod{2}. \end{cases}$$

For $k \equiv 0 \pmod{2}$, let L_k be defined by four generating vectors as follows:

$$\begin{aligned} \mathbf{v}_1 &= (-a - 1, a + 1, a, -a + 1) \\ \mathbf{v}_2 &= (a - 1, a + 1, a + 1, -a) \\ \mathbf{v}_3 &= (-a - 1, -a + 1, a + 1, -a) \\ \mathbf{v}_4 &= (-a, -a, a, a + 1). \end{aligned}$$

Then the following vectors are in L_k :

$$\begin{aligned} -(2a^2 + 2a + 1)\mathbf{v}_1 + (2a^2 + a + 2)\mathbf{v}_2 - (a + 2)\mathbf{v}_3 + \mathbf{v}_4 &= (4a^3 + 4a^2 + 6a + 1, -1, 0, 0), \\ -(2a^3 - 1)\mathbf{v}_1 + (2a^3 - a^2 + 2a - 2)\mathbf{v}_2 - (a^2 + a - 1)\mathbf{v}_3 + (a - 1)\mathbf{v}_4 &= (4a^4 + 4a^2 - 4a, 0, -1, 0), \\ -2a^3\mathbf{v}_1 + (2a^3 - a^2 + 2a - 1)\mathbf{v}_2 - (a^2 + a - 1)\mathbf{v}_3 + (a - 1)\mathbf{v}_4 &= (4a^4 + 4a^2 - 2a, 0, 0, -1). \end{aligned}$$

Hence, we have $\mathbf{e}_2 = (4a^3 + 4a^2 + 6a + 1)\mathbf{e}_1$, $\mathbf{e}_3 = (4a^4 + 4a^2 - 4a)\mathbf{e}_1$ and

$\mathbf{e}_4 = (4a^4 + 4a^2 - 2a)\mathbf{e}_1$ in \mathbb{Z}^4/L_k , and so \mathbf{e}_1 generates \mathbb{Z}^4/L_k .

$$\text{Also, } \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 + 8a^3 + 12a^2 + 4a & 0 & 0 & 0 \\ 4a^3 + 4a^2 + 6a + 1 & -1 & 0 & 0 \\ 4a^4 + 4a^2 - 4a & 0 & -1 & 0 \\ 4a^4 + 4a^2 - 2a & 0 & 0 & -1 \end{pmatrix}$$

$= -(8a^4 + 8a^3 + 12a^2 + 4a) = -(k^4 + 2k^3 + 6k^2 + 4k)/2 = -LK_{circ}(8, k)$, as in the statement of the theorem.

Thus, \mathbb{Z}^4/L_k is isomorphic to $\mathbb{Z}_{LK_{circ}(8,k)}$ via an isomorphism taking $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ to 1, $4a^3 + 4a^2 + 6a + 1, 4a^4 + 4a^2 - 4a, 4a^4 + 4a^2 - 2a$. As $a = k/2$ this gives the first generating set specified in the theorem:

$$\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k)/4\}.$$

Similarly, for $k \equiv 1 \pmod{2}$ let L_k be defined by four generating vectors as follows:

$$\begin{aligned}\mathbf{v}_1 &= (-a+1, a+1, -a+1, a) \\ \mathbf{v}_2 &= (a+1, a+1, -a+2, a-1) \\ \mathbf{v}_3 &= (-a-1, a-1, a-1, -a) \\ \mathbf{v}_4 &= (-a, a, a, a-1).\end{aligned}$$

In this case, the following vectors are in L_k :

$$\begin{aligned}&-(2a^2 + a + 2)\mathbf{v}_1 + (2a^2 + 2a + 1)\mathbf{v}_2 - a\mathbf{v}_3 - \mathbf{v}_4 = (4a^3 - 4a^2 + 6a - 1, -1, 0, 0), \\ &-(2a^3 - a^2 - 2a - 2)\mathbf{v}_1 + (2a^3 - 4a - 1)\mathbf{v}_2 - (a^2 - a - 1)\mathbf{v}_3 - (a - 1)\mathbf{v}_4 = \\ &(4a^4 - 8a^3 + 8a^2 - 8a, 0, -1, 0), \\ &-(2a^3 - a^2 - 2a - 1)\mathbf{v}_1 + (2a^3 - 4a)\mathbf{v}_2 - (a^2 - a - 1)\mathbf{v}_3 - (a - 1)\mathbf{v}_4 = \\ &(4a^4 - 8a^3 + 8a^2 - 6a, 0, 0, -1).\end{aligned}$$

Hence, we have $\mathbf{e}_2 = (4a^3 + 4a^2 + 6a - 1)\mathbf{e}_1$, $\mathbf{e}_3 = (4a^4 - 8a^3 + 8a^2 - 8a)\mathbf{e}_1$ and $\mathbf{e}_4 = (4a^4 - 8a^3 + 8a^2 - 6a)\mathbf{e}_1$, in \mathbb{Z}^4/L_k , and so \mathbf{e}_1 generates \mathbb{Z}^4/L_k .

$$\text{Also, } \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 - 8a^3 + 12a^2 - 4a & 0 & 0 & 0 \\ 4a^3 - 4a^2 + 6a - 1 & -1 & 0 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 8a & 0 & -1 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 6a & 0 & 0 & -1 \end{pmatrix}$$

$= -(8a^4 - 8a^3 + 12a^2 - 4a) = -(k^4 + 2k^3 + 6k^2 + 6k + 1)/2 = -LK_{\text{circ}}(8, k)$, as in the statement of the theorem.

Thus, \mathbb{Z}^4/L_k is isomorphic to $\mathbb{Z}_{LK_{\text{circ}}(8, k)}$ with generators

$1, 4a^3 - 4a^2 + 6a - 1, 4a^4 - 8a^3 + 8a^2 - 8a, 4a^4 - 8a^3 + 8a^2 - 6a$. As $a = (k+1)/2$ in this case, this gives the second generating set specified in the theorem:
 $\{1, (k^3 + k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}$.

It remains to show that $S_{f, k} + L_k = \mathbb{Z}^4$. First, we consider the case $k \equiv 0 \pmod{2}$.

For $k = 2$, it is straightforward to show directly that \mathbb{Z}_{32} with generators $1, 4, 6, 15$ has diameter 2. So we assume $k \geq 4$, giving $a \geq 2$. Let

$$\begin{aligned}\mathbf{v}_5 &= \mathbf{v}_1 - \mathbf{v}_3 + \mathbf{v}_4 = (-a, a, a-1, a+2) \\ \mathbf{v}_6 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = (-a, a, -a-1, -a) \\ \mathbf{v}_7 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = (-a+1, a-1, -a-2, a+1) \\ \mathbf{v}_8 &= \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = (a, a, a, a+1)\end{aligned}$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as defined for $k \equiv 0 \pmod{2}$. Then the 16 vectors $\pm \mathbf{v}_i$ for $i = 1, \dots, 8$ provide one element of L_k lying strictly within each of the 16 orthants of

\mathbb{Z}^4 . Most of the coordinates of these vectors have absolute value at most $a + 1$. Only $\pm \mathbf{v}_5$ and $\pm \mathbf{v}_7$ each have one coordinate with absolute value equal to $a + 2$.

Now we consider the case $k \equiv 1 \pmod{2}$. For $k = 3$, it may be shown directly that \mathbb{Z}_{104} with generators 1, 16, 20, 27 has diameter 3. So we assume $k \geq 5$, giving $a \geq 3$, and let

$$\begin{aligned}\mathbf{v}_5 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = (-a, -a, -a - 1, -a + 2) \\ \mathbf{v}_6 &= \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = (a, a, -a + 1, -a) \\ \mathbf{v}_7 &= \mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_4 = (-a, a, -a, -a + 1) \\ \mathbf{v}_8 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = (-a + 1, -a + 1, -a, a + 1)\end{aligned}$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as defined for $k \equiv 1 \pmod{2}$, so that the 16 vectors $\pm \mathbf{v}_i$ provide one element of \mathbf{L}_k lying strictly within each of the orthants of \mathbb{Z}^4 . In this case, all the coordinates of these vectors have absolute value at most $a + 1$.

We must show that each $\mathbf{x} \in \mathbb{Z}^4$ is in $S_{f,k} + L_k$, which means that for any $\mathbf{x} \in \mathbb{Z}^4$ we need to find a $\mathbf{w} \in L_k$ such that $\mathbf{x} - \mathbf{w} \in S_{f,k}$. However, $\mathbf{x} - \mathbf{w} \in S_{f,k}$ if and only if $\delta(\mathbf{x}, \mathbf{w}) \leq k$, where δ is the Manhattan norm on \mathbb{Z}^4 . If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^4$ and each coordinate of \mathbf{y} lies between the corresponding coordinate of \mathbf{x} and \mathbf{z} or is equal to one of them, then $\delta(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{y}, \mathbf{z}) = \delta(\mathbf{x}, \mathbf{z})$. In such a case we say that “ \mathbf{y} lies between \mathbf{x} and \mathbf{z} ”.

For any $\mathbf{x} \in \mathbb{Z}^4$, we reduce \mathbf{x} by adding appropriate elements of L_k until the resulting vector lies within Manhattan distance k of $\mathbf{0}$ or some other element of L_k . The first stage is to reduce \mathbf{x} to a vector whose coordinates all have absolute value at most $a + 1$. If \mathbf{x} has a coordinate with absolute value above $a + 1$, then let \mathbf{v} be one of the vectors $\pm \mathbf{v}_i$ ($1 \leq i \leq 8$) such that the coordinates of \mathbf{v} have the same sign as the corresponding coordinates of \mathbf{x} . If a coordinate of \mathbf{x} is 0, then either sign is allowed for \mathbf{v} as long as the corresponding coordinate of \mathbf{v} has absolute value $\leq a + 1$. So in the case $k \equiv 0 \pmod{2}$, if the \mathbf{e}_3 coordinate of \mathbf{x} is 0 then we avoid \mathbf{v}_7 and take \mathbf{v}_5 instead. Also, if the \mathbf{e}_4 coordinate of \mathbf{x} is 0 (or both \mathbf{e}_3 and \mathbf{e}_4 coordinates are 0) then instead of \mathbf{v}_5 we take \mathbf{v}_1 .

Now consider $\mathbf{x}' = \mathbf{x} - \mathbf{v}$. If a coordinate of \mathbf{x} has absolute value s , $1 \leq s \leq a + 1$, then the corresponding coordinate of \mathbf{x}' will have absolute value $s' \leq a + 1$ because of the sign matching and the fact that the coordinates of \mathbf{v} have absolute value $\leq a + 2$. If a coordinate of \mathbf{x} has absolute value $s = 0$, then as indicated above, the corresponding value of \mathbf{x}' will have absolute value $s' \leq a + 1$ because \mathbf{v} is chosen such that the corresponding coordinate has absolute value $\leq a + 1$. If a coordinate of \mathbf{x} has absolute value $s > a + 1$, then the corresponding coordinate of \mathbf{x}' will be strictly smaller in

absolute value. Therefore, repeating this procedure will result in a vector whose coordinates all have absolute value at most $a + 1$.

If the resulting vector \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v} , where $\mathbf{v} = \pm \mathbf{v}_i$ for some i , then we have $\delta(\mathbf{0}, \mathbf{x}') + \delta(\mathbf{x}', \mathbf{v}) = \delta(\mathbf{0}, \mathbf{v})$. For $k \equiv 0 \pmod{2}$, all of the vectors \mathbf{v} satisfy $\delta(\mathbf{0}, \mathbf{v}) = 4a + 1$, and for $k \equiv 1 \pmod{2}$ they all satisfy $\delta(\mathbf{0}, \mathbf{v}) = 4a - 1$. So in either case we have $\delta(\mathbf{0}, \mathbf{v}) = 2k + 1$. Since $\delta(\mathbf{0}, \mathbf{x}')$ and $\delta(\mathbf{x}', \mathbf{v})$ are both non-negative integers, one of them must be at most k , so that $\mathbf{x}' \in S_{f,k} + L_k$. Hence, we also have $\mathbf{x} \in S_{f,k} + L_k$ as required.

We are left with the case where the absolute value of each coordinate of the reduced \mathbf{x} is at most $a + 1$, and \mathbf{x} is in the orthant of \mathbf{v} , where $\mathbf{v} = \pm \mathbf{v}_i$ for some $i \leq 8$ but does not lie between $\mathbf{0}$ and \mathbf{v} . Since L_k is centrosymmetric, we only need to consider the eight orthants containing $\mathbf{v}_1, \dots, \mathbf{v}_8$. For both cases $k \equiv 0$ and $k \equiv 1 \pmod{2}$, the exceptions need to be considered for each orthant in turn. Only the exceptions for the orthant of \mathbf{v}_1 for $k \equiv 0 \pmod{2}$ and for $k \equiv 1 \pmod{2}$ are included here. The other orthants are handled similarly. A full proof including all orthants for both cases is available on arXiv [27].

So suppose that $k \equiv 0 \pmod{2}$ and \mathbf{x} lies within the orthant of \mathbf{v}_1 , but not between $\mathbf{0}$ and \mathbf{v}_1 . Then as $\mathbf{v}_1 = (-a - 1, a + 1, a, -a + 1)$, the third coordinate of \mathbf{x} is equal to $a + 1$ or the fourth coordinate equals $-a$ or $-a - 1$. We now distinguish three cases.

Case 1: $\mathbf{x} = (-r, s, a + 1, -u)$ where $0 \leq r, s \leq a + 1$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, 1, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r \leq 1$ or $s \leq 1$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (2 - r, s - 2, -a - 1, 2a - u)$. If $r \leq 1$ and $s \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (1 - a - r, a - 1 + s, -1, a + 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 . If $r \leq 1$ and $s \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r \geq 2$ and $s \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 2: $\mathbf{x} = (-r, s, a + 1, -u)$ where $0 \leq r, s \leq a + 1$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, 1, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = 0$ or $s = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (1 - r, s - 1, -a, -u - 1)$. If $r = 0$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $r = 0$ and $s \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-r, s, t, -u)$ where $0 \leq r, s \leq a + 1$ and $0 \leq t \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, t - a, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $s = 0$ or $t = 0$. If $r = 0$ and $s = 0$, then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_7$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1 - r, s - 1, t - 1, 2a + 1 - u)$. If $r = 0, s \geq 1$ and $t \geq 1$ then \mathbf{x}'' lies

between $\mathbf{0}$ and \mathbf{v}_8 . Let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_4 = (-a - r, s - a, a + t, a + 1 - u)$. If $r = 0$ and $s \geq 1$ and $t = 0$, then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_4 unless $s = a + 1$, in which case if $u = a$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_2 , and if $u = a + 1$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_4 . Let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, a - 1 + s, t - a - 1, a - u)$. If $r \geq 1, s = 0$ and $t \geq 1$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1, s = 0$ and $t = 0$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_3$ if $u = a$, and between $\mathbf{0}$ and \mathbf{v}_6 if $u = a + 1$. If $r \geq 1, s \geq 1$ and $t = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 unless $r = a + 1$ or $s = a + 1$. If $r = a + 1, s \geq 1$ and $t = 0$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r \geq 1, s = a + 1$ and $t = 0$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

This completes the cases for the orthant of \mathbf{v}_1 for $k \equiv 0 \pmod{2}$.

Now suppose that $k \equiv 1 \pmod{2}$ and \mathbf{x} lies within the orthant of \mathbf{v}_1 , but not between $\mathbf{0}$ and \mathbf{v}_1 . Then the first coordinate of \mathbf{x} is equal to $-a$ or $-a - 1$, or the third coordinate equals $-a$ or $-a - 1$, or the fourth equals $a + 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, -t, a + 1)$ where $a \leq r, t \leq a + 1$ and $0 \leq s \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $s \leq 1$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (2a - 2 - r, s - 2, 2a - 1 - t, -a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 2: $\mathbf{x} = (-r, s, -t, u)$ where $a \leq r, t \leq a + 1$ and $0 \leq s \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $s = 0$ or $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (2a - 1 - r, s - 1, 2a - t, u - 2)$. If $s = 0$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$, unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (a - r, a, 1, u + a - 2)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 . If $s = 0$ and $u \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $s \geq 1$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a + 1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_8 = (a - r, 1, a - t, a + u - 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 .

Case 3: $\mathbf{x} = (-r, s, -t, a + 1)$ where $a \leq r \leq a + 1, 0 \leq s \leq a + 1$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (2a - 1 - r, -1, -t, -a + 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 4: $\mathbf{x} = (-r, s, -t, a + 1)$ where $0 \leq r \leq a - 1, 0 \leq s \leq a + 1$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-2 - r, s - 2, 2a - 2 - t, -a + 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, -t, a + 1)$ where $0 \leq r, t \leq a - 1$ and $0 \leq s \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless

$s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (-r - 1, -1, -t - 1, -a + 2)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 6: $\mathbf{x} = (-r, s, -t, u)$ where $0 \leq r \leq a - 1$, $0 \leq s \leq a + 1$, $a \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (-r - 1, s - 1, 2a - 1 - t, u - 1)$. If $s = 0$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a - r, a, a + 1 - t, a - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $s = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $s \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = a + 1$, in which case \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 7: $\mathbf{x} = (-r, s, -t, u)$ where $a \leq r \leq a + 1$, $0 \leq s \leq a + 1$, $0 \leq t \leq a - 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (2a - r, s, -t + 1, u - 1)$. If $t = 0$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a + 1 - r, s - a + 1, -a + 1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $a \leq s \leq a + 1$, in which case let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_7 = (a - r, s - a, a, a - 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 . If $t = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a + 1$ or $u = a$ in which case let $\mathbf{x}^v = \mathbf{x}'' + \mathbf{v}_5 = (a - r, s - a, -a, -a + u + 1)$. If $s = a + 1$ then \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_3 . If $1 \leq s \leq a$ and $u = a$ then \mathbf{x}^v lies between $\mathbf{0}$ and \mathbf{v}_8 . If $s = 0$ and $u = a$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $t \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_6 unless $s = a + 1$, in which case \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_3 .

This completes the cases for the orthant of \mathbf{v}_1 for $k \equiv 1 \pmod{2}$. □

4.5 Lattice generating vectors for the degree 10 families

An outline proof of the existence of the largest-known degree 10 circulant graph family for diameter class 0, $k \equiv 0 \pmod{5}$, was published in 2018 [26], and a more complete version covering all diameter classes can be found on arXiv [28]. However, resolution of the residual boundary exceptions was only included for the first orthant of the solution space in \mathbb{Z}^5 for diameter class 0, as an example, as the full set of boundary exception resolution for all orthants and all diameter classes runs to several thousand pages. They were all resolved and confirmed using a tailored computer program, and the output is available electronically for inspection.

As the format of the degree 10 existence proof is the same as the degree 8 proof presented in the previous section 4.4, we only include a set of lattice generating vectors for all diameter classes, as these lie at the core of the proofs, see Table 4.11.

Table 4.11: Lattice generating vectors for largest-known degree 10 circulant graph families

Diameter class (mod 5)	Parameter value	One set of lattice generating vectors
$k \equiv 0$	$a = 2k/5$	$\mathbf{v}_1 = (a - 1, -a - 2, -a, -a - 1, -a + 1)$ $\mathbf{v}_2 = (a, a - 1, -a - 1, -a - 1, -a)$ $\mathbf{v}_3 = (a, -a - 1, a, -a, -a)$ $\mathbf{v}_4 = (a, -a - 2, -a - 1, a - 1, -a + 1)$ $\mathbf{v}_5 = (a - 1, -a - 1, -a + 1, -a - 1, a + 1)$
$k \equiv 1$	$a = (2k + 3)/5$	$\mathbf{v}_1 = (a - 1, -a - 1, -a - 1, -a + 1, -a + 2)$ $\mathbf{v}_2 = (a, a - 2, -a - 1, -a, -a + 1)$ $\mathbf{v}_3 = (a, -a - 1, a - 2, -a, -a + 1)$ $\mathbf{v}_4 = (a - 1, -a, -a, a, -a + 1)$ $\mathbf{v}_5 = (a - 1, -a, -a, -a + 2, a + 1)$
$k \equiv 2$	$a = (2k + 1)/5$	$\mathbf{v}_1 = (a - 2, -a - 1, -a, -a - 1, -a)$ $\mathbf{v}_2 = (a - 1, a - 1, -a - 1, -a - 1, -a)$ $\mathbf{v}_3 = (a - 1, -a, a, -a - 1, -a)$ $\mathbf{v}_4 = (a - 1, -a - 1, -a, a - 1, -a - 1)$ $\mathbf{v}_5 = (a - 1, -a - 1, -a, -a, a)$
$k \equiv 3$	$a = (2k - 1)/5$	$\mathbf{v}_1 = (a - 1, -a - 2, -a - 2, -a, -a + 1)$ $\mathbf{v}_2 = (a, a - 1, -a - 2, -a - 1, -a)$ $\mathbf{v}_3 = (a, -a - 2, a - 1, -a - 1, -a)$ $\mathbf{v}_4 = (a - 1, -a - 1, -a - 1, a + 1, -a)$ $\mathbf{v}_5 = (a - 1, -a - 1, -a - 1, -a + 1, a + 2)$
$k \equiv 4$	$a = (2k + 2)/5$	$\mathbf{v}_1 = (a - 2, -a - 1, -a, -a - 1, -a + 1)$ $\mathbf{v}_2 = (a - 1, a - 2, -a - 1, -a - 1, -a)$ $\mathbf{v}_3 = (a - 1, -a, a, -a, -a)$ $\mathbf{v}_4 = (a - 1, -a - 1, -a - 1, a - 1, -a + 1)$ $\mathbf{v}_5 = (a - 2, -a, -a + 1, -a - 1, a + 1)$

These sets of lattice generating vectors can be taken as the rows of 5×5 matrices, forming lattice generator matrices (LGMs) for each graph family. An associated matrix, the LGM odd basis for each largest-known circulant graph family of degree 10, along with LGMs for each largest-known degree 11 circulant graph family, are presented in Appendix A.4. Lattice generator matrices are introduced and discussed in detail in Chapter 6.

PROPERTIES OF ABELIAN CAYLEY GRAPH FAMILIES

In this chapter, we discuss some important properties of Abelian Cayley graph families. A common property of Abelian Cayley graphs relates to their distance partition profiles. We will explore distance levels within the profile and when they may be considered maximal. The prime motivation is to determine if these profiles have regular structure related to extremal and largest-known families that might assist in the search for extremal graph families of higher degree. Vertices in an Abelian Cayley graph may be characterised not only by their distance from an arbitrary reference vertex but also by the number of different connection set elements that generate the edges in a shortest path to each vertex. Distance partition profiles that include this classification are also investigated. The relation between graph families being quasimaximal and having maximum odd girth is introduced. Two essential relationships between graph families are also presented: conjugation, which relates two quasimaximal families of the same degree; and translation, between an odd-degree family and an even-degree family of the same odd dimension. These relationships are treated in more depth in Chapter 6.

5.1 Distance partition profiles

Before discussing distance partition profiles, some definitions are introduced. A graph is *distance transitive* if, given any two ordered pairs of vertices (u, u') and (v, v') such that $\text{distance}(u, u') = \text{distance}(v, v')$, there is an automorphism mapping (u, u') to (v, v') . If u is a vertex of a graph X of diameter k , then let $X_i(u)$ denote the set of vertices at distance i from u . The partition $\{\{u\}, X_1(u), \dots, X_k(u)\}$ is called the *distance partition* with respect to u . For vertex-transitive graphs, the cardinality of $X_i(u)$ for any i is independent of u . This is the basis for a related formal definition of a *distance partition profile*.

Definition 5.1. For any vertex-transitive graph X of diameter k , the *distance partition profile* of X is a vector of length $k + 1$ whose i -th coordinate ($0 \leq i \leq k$) is the cardinality of the element $X_i(u)$ of its distance partition with respect to any vertex, u .

We now adapt a property defined for distance-transitive graphs to be applicable to vertex-transitive graphs. For a distance-transitive graph of diameter k , each vertex in any given level of its distance partition is joined to a common number of vertices in the level below, a common number in the same level, and a common number in the level above. These three sets of $k + 1$ numbers form a $3 \times (k + 1)$ array called the *intersection array* of the graph. In general, a vertex-transitive graph does not have an intersection array because the vertices within each level of its distance partition do not all have the same number of edges to vertices in the three neighbouring levels. However, we can define a variant that counts the total number of edges within and between distance levels.

Definition 5.2. Let X be a vertex-transitive graph of diameter k . The *total intersection array* has the same $3 \times (k + 1)$ format as a standard intersection array, but each element counts the total number of adjacent vertices summed across all vertices within each level of the distance partition.

For extremal and largest-known graph families up to degree 9, we will explore their distance partition profiles and the manner in which the size of each distance level increases with increasing diameter until it reaches a maximum value. We will discover that the number of levels at their maximum value depends linearly on the diameter for any given degree. These observed relationships will be proved in Chapter 6.

For extremal circulant graphs of dimension 1, for degree 2 the distance partition profiles for increasing diameter k are $(1, 2), (1, 2, 2), (1, 2, 2, 2), (1, 2, 2, 2, 2), \dots$; and for degree 3 are $(1, 3), (1, 3, 4), (1, 3, 4, 4), (1, 3, 4, 4, 4), \dots$. In both cases, the size of each successive level is a constant (2 or 4 respectively). For extremal circulant graphs of dimension 2, we find for degree 4 the sequence $(1, 4), (1, 4, 8), (1, 4, 8, 12), (1, 4, 8, 12, 16)$, etc, so that each successive level is 4 more than the previous. For degree 5, each new level does not immediately take its final size: from level 2 onwards the size of each new level is 2 below its maximum value, which it reaches when the next level is added. See Table 5.1. From level 3 onwards, the

Table 5.1: Distance partition profiles for extremal circulant graphs of degree 5

Diameter k	Order $Ext_{circ}(5, k)$	Distance partition level							
		0	1	2	3	4	5	6	
1	6	1	5						
2	16	1	5	10					
3	36	1	5	12	18				
4	64	1	5	12	20	26			
5	100	1	5	12	20	28	34		
6	144	1	5	12	20	28	36	42	

maximum size of each successive level is 8 more than the previous. So for both degree 3 and 4, the size of the levels increase at a constant rate.

For dimension 3, the evolution of the distance partition profiles for increasing diameter k becomes more complicated. The profiles for the largest-known graphs of degree 6 for diameter $k \leq 15$ are shown in Table 5.2. For diameters with two isomorphism classes, both have the same distance partition profile.

Table 5.2: Distance partition profiles for largest-known circulant graphs of degree 6

Diameter k	Order $LK_{circ}(6, k)$	Distance partition level															
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	7	1	6														
2	21	1	6	14													
3	55	1	6	18	30												
4	117	1	6	18	38	54											
5	203	1	6	18	38	62	78										
6	333	1	6	18	38	66	94	110									
7	515	1	6	18	38	66	102	134	150								
8	737	1	6	18	38	66	102	142	174	190							
9	1027	1	6	18	38	66	102	146	190	222	238						
10	1393	1	6	18	38	66	102	146	198	246	278	294					
11	1815	1	6	18	38	66	102	146	198	254	302	334	350				
12	2329	1	6	18	38	66	102	146	198	258	318	366	398	414			
13	2943	1	6	18	38	66	102	146	198	258	326	390	438	470	486		
14	3629	1	6	18	38	66	102	146	198	258	326	398	462	510	542	558	
15	4431	1	6	18	38	66	102	146	198	258	326	402	478	542	590	622	638

As for degree 5, the size of each new level is initially below its maximum value, but now the number of increments to reach its maximum is not fixed at 1 but increases with increasing diameter, so that the maximal zone, where the levels have reached their maximum value, covers about the first two thirds of the levels. Also, the difference between the maximum size of successive levels is no longer constant but increases linearly. Degree 7 is similar, with distance partition profile independent of isomorphism class and maximal zone covering two thirds of the levels. For the largest-known circulant graphs of dimension 4, the evolution of the distance partition profiles follows a similar structure but with certain differences. For both degree 8 and 9, the maximal zone covers about the first half of the levels, and the difference between the maximum size of successive levels increases as a quadratic. For degree 9, the two isomorphism classes for odd diameter share the same profile. We note that for dimensions $f = 2$ to 4, the proportion of the distance levels covered by the maximal zone is about $2/f$. In Section 6.6, this proportion is proved to remain valid for all higher dimensions, see Theorem 6.9.

The use of distance partitions in the analysis of extremal graphs can be taken a stage further by defining an extension of the intersection array that can be applied to vertex-transitive graphs, called the *total intersection array*. So taking Godsil and Royle's example of the dodecahedron [16] which has distance partition profile $(1, 3, 6, 6, 3, 1)$ and standard intersection array

$$\begin{pmatrix} - & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & - \end{pmatrix}, \text{ its total intersection array becomes } \begin{pmatrix} - & 3 & 6 & 6 & 6 & 3 \\ 0 & 0 & 6 & 6 & 0 & 0 \\ 3 & 6 & 6 & 6 & 3 & - \end{pmatrix}.$$

With this definition, the sum of the elements in each column of the total intersection array is equal to the corresponding element of the distance partition profile multiplied by the degree. Total intersection arrays provide a useful view on the structure of the graphs. The position of the first non-zero element in the middle row determines the odd girth of the graph, and if all those elements are zero then the graph is bipartite. They can also distinguish between non-isomorphic graphs of common degree, diameter and order that might have the same distance partition profile. An example is provided by the four isomorphism classes of extremal circulant graphs of degree 9 and diameter 3, which all have the same odd girth and distance partition profile. These are easily proved to be distinct by determining their total intersection arrays, which are all different; see Table 5.3.

Table 5.3: Extremal circulant graphs of the four isomorphism classes for degree 9, diameter 3

Diameter, k	Order, $Ext_{circ}(9, 3)$	Generating set	Odd girth	Distance partition profile	Total intersection array
3	130	$\{1, 8, 14, 47\}$	5	$(1, 9, 40, 80)$	$\begin{pmatrix} - & 9 & 72 & 244 \\ 0 & 0 & 44 & 476 \\ 9 & 72 & 244 & - \end{pmatrix}$
3	130	$\{1, 8, 20, 35\}$	5	$(1, 9, 40, 80)$	$\begin{pmatrix} - & 9 & 72 & 242 \\ 0 & 0 & 46 & 478 \\ 9 & 72 & 242 & - \end{pmatrix}$
3	130	$\{1, 26, 49, 61\}$	5	$(1, 9, 40, 80)$	$\begin{pmatrix} - & 9 & 72 & 286 \\ 0 & 0 & 2 & 434 \\ 9 & 72 & 286 & - \end{pmatrix}$
3	130	$\{2, 8, 13, 32\}$	5	$(1, 9, 40, 80)$	$\begin{pmatrix} - & 9 & 72 & 234 \\ 0 & 0 & 54 & 486 \\ 9 & 72 & 234 & - \end{pmatrix}$

Reverting to the discussion on distance partition profiles, some obvious questions arise.

1. What is the structure behind the maximum size of each level?
2. What is the logic behind the evolution of the size of each level until it reaches its maximum?
3. What determines the number of levels in the maximal zone?

First we define a derived upper bound for the size of a level in the distance partition of an Abelian Cayley graph.

Definition 5.3. For any Abelian Cayley graph of degree $d \geq 2$ and diameter $k \geq 2$, the *derived upper bound* for the size of level l in its distance partition for $2 \leq l \leq k$, $Upp_{AbCay}^{level}(d, l)$, is determined by the first order difference of the Abelian Cayley upper bound $Upp_{AbCay}(d, l)$ described in Section 2.2 for increasing l , so that $Upp_{AbCay}^{level}(d, l) = Upp_{AbCay}(d, l) - Upp_{AbCay}(d, l - 1)$.

Values for $d \leq 11$ and $l \leq 10$ are shown in Table 5.4.

Table 5.4: $Upp_{AbCay}^{level}(d, l)$, first order difference of the upper bound, $Upp_{AbCay}(d, l)$

Dimension	Degree d	Distance, l , from the reference vertex at level 0								
		2	3	4	5	6	7	8	9	10
1	2	2	2	2	2	2	2	2	2	2
	3	4	4	4	4	4	4	4	4	4
2	4	8	12	16	20	24	28	32	36	40
	5	12	20	28	36	44	52	60	68	76
3	6	18	38	66	102	146	198	258	326	402
	7	24	56	104	168	248	344	456	584	728
4	8	32	88	192	360	608	952	1408	1992	2720
	9	40	120	280	552	968	1560	2360	3400	4712
5	10	50	170	450	1002	1970	3530	5890	9290	14002
	11	60	220	620	1452	2972	5500	9420	15180	23292

These values are precisely the maximum size of each corresponding level of the distance partitions. So over the range of degrees and diameters considered, we can see that for each extremal and largest-known circulant graph each distance partition level l is filled to the maximum determined by the upper bound $Upp_{AbCay}(d, l)$. This is an immediate consequence of the corresponding lattice covering of \mathbb{Z}^f . Lee spheres of radius k , the diameter of the graph, provide a complete covering, with overlapping that increases with dimension. For any given dimension and diameter, there will be a critical value l^* such that Lee spheres of radius $l \leq l^*$ centred on the lattice do not intersect each other. These values of l correspond to the maximal distance levels of the graph. Formulae for $Upp_{AbCay}^{level}(d, l)$ as a function of l are presented for degree 2 to

11 in Table 5.5. For the graphs of dimension 1 and 2, all the levels are maximal, with the exception of the last level for degree 5. For dimension 3, the proportion of maximal levels is about $2/3$, for dimension 4 about $1/2$ and for dimension 5 about $2/5$. The exact number of maximal levels in each case is also shown in Table 5.5. For $f \geq 2$, we note that these proportions are represented by the expression $2/f$. The general validity of this ratio is proved in Chapter 6.

Table 5.5: Maximal distance partition levels of extremal and largest-known circulant graphs of degree 2 to 11 and diameter k : size of each maximal level, $Upp_{AbCay}^{level}(d, l)$, and the position of the last maximal level

Dimension f	Degree d	$Upp_{AbCay}^{level}(d, 1)$	$Upp_{AbCay}^{level}(d, l), l \geq 2$	Last maximal level
1	2	2	2	k
	3	3	4	k
2	4	4	$4l$	k
	5	5	$8l - 4$	$k - 1$
3	6	6	$4l^2 + 2$	$\lfloor (2k + 1)/3 \rfloor$
	7	7	$8l^2 - 8l + 8$	$\lfloor 2k/3 \rfloor$
4	8	8	$(8l^3 + 16l)/3$	$\lfloor (k + 1)/2 \rfloor$
	9	9	$(16l^3 - 24l^2 + 56l - 24)/3$	$\lfloor k/2 \rfloor$
5	10	10	$(4l^4 + 20l^2 + 6)/3$	$\lfloor (2k + 3)/5 \rfloor$
	11	11	$(8l^4 - 16l^3 + 64l^2 - 56l + 36)/3$	$\lfloor (2k + 2)/5 \rfloor$

5.2 Distance partition profile by vertex type

We have seen that all the extremal and largest-known circulant graphs of degree 2 to 9 of arbitrary diameter k , above some threshold, have maximum odd girth, $2k + 1$. This means that only in the final distance partition level, k , relative to an arbitrary root vertex, is any vertex adjacent to another in the same level. Thus, any vertex in level l for $1 \leq l \leq k - 1$ is adjacent only to vertices in level $l - 1$ or $l + 1$ and to none in level l . In this sense, such a vertex may be defined as *thin*. As the degree, d , of each vertex is fixed, if it is adjacent to s vertices in level $l - 1$ then it must be adjacent to $d - s$ in level $l + 1$, and such a thin vertex is defined to be of type T_s . Vertices in level k may be adjacent to others in the same level but not of course to any in a further level. Therefore, the type of these thin vertices is also well-defined by the number of adjacent vertices in the preceding level.

Analysis of the extremal and largest-known circulant graphs of degree 2 to 9 reveals a regular structure in the number of vertices of each type in each distance partition level. Examples for graphs of degree 4, 6 and 8, all with diameter 12, are shown in

Tables 5.6, 5.7 and 5.8, including the successive differences of the sequences at appropriate order s , Δ^s , in the maximal and submaximal zones.

Definition 5.4. The successive difference of order s , Δ^s , of a sequence S , is defined as follows. A first order difference sequence is created by taking the difference between successive members of sequence S . For any n , an n th-order sequence is created by taking the difference between successive members of the $(n - 1)$ th-order sequence. If there is an s such that the members of the s th-order sequence are a constant value $c > 0$, say, then the original sequence S is defined to have an s th-order difference of c , and we write $\Delta^s = c$. In case the members of S have a constant value c , then we define $\Delta^0 = c$.

Table 5.6: Distance partition profile by vertex type: extremal graph of degree 4 and diameter 12

Vertex type	Distance partition level													Differences	
	0	1	2	3	4	5	6	7	8	9	10	11	12	Maximal	
T_0	1														
T_1		4	4	4	4	4	4	4	4	4	4	4	4	$\Delta^0 = 4$	
T_2			4	8	12	16	20	24	28	32	36	40	44	$\Delta^1 = 4$	
Total	1	4	8	12	16	20	24	28	32	36	40	44	48	$\Delta^1 = 4$	

Table 5.7: Distance partition profile by vertex type: largest-known graph of degree 6, diameter 12

Vertex type	Distance partition level													Differences	
	0	1	2	3	4	5	6	7	8	9	10	11	12	Maximal	Submaximal
T_0	1														
T_1		6	6	6	6	6	6	6	6					$\Delta^0 = 6$	$\Delta^0 = 0$
T_2			12	24	36	48	60	72	84	88	76	64	52	$\Delta^1 = 12$	$\Delta^1 = -12$
T_3				8	24	48	80	120	168	226	274	306	322	$\Delta^2 = 8$	$\Delta^2 = -16$
T_4										4	16	28	40		$\Delta^1 = 12$
Total	1	6	18	38	66	102	146	198	258	318	366	398	414	$\Delta^2 = 8$	$\Delta^2 = -16$

Table 5.8: Distance partition profile by vertex type: largest-known graph of degree 8, diameter 12

Vertex type	Distance partition level													Differences	
	0	1	2	3	4	5	6	7	8	9	10	11	12	Maximal	Submaximal
T_0	1														
T_1		8	8	8	8	8	8	4						$\Delta^0 = 8$	$\Delta^0 = 0$
T_2			24	48	72	96	120	136	124	100	76	52	26	$\Delta^1 = 24$	$\Delta^1 = -24$
T_3				32	96	192	320	476	624	720	752	720	624	$\Delta^2 = 32$	$\Delta^2 = -64$
T_4					16	64	160	328	564	844	1124	1356	1495	$\Delta^3 = 16$	$\Delta^3 = -48$
T_5									32	96	192	320	476		$\Delta^2 = 32$
T_6													2		
Total	1	8	32	88	192	360	608	944	1344	1760	2144	2448	2623	$\Delta^3 = 16$	$\Delta^3 = -48$

In these examples for graphs of even degree, we can see that the number of different vertex types increases with degree. Graphs of every degree contain type T_1 vertices, with their number remaining constant and equal to the degree within the maximal zone, while becoming absent from the submaximal zone within the first few levels. The number of each subsequent vertex type grows with a constant higher order difference in the maximal zone before reversing with a constant difference of the same order in the submaximal zone. In each case, the values of these constants depend only on the degree and are independent of diameter. The graphs of odd degree display a similar structure. Common parameters of all the distance partition profiles by vertex type are presented in Table 5.9 for the maximal levels and in Table 5.10 for the submaximal levels.

Table 5.9: Number of vertices of each type within maximal levels of extremal and largest-known graphs of degrees 4 to 9 (dimension $f = 2$ to 4)

Vertex type T_s	Even degree $d = 2f$		Odd degree $d = 2f + 1$	
	Difference order	Value	Difference order	Value
$1 \leq s \leq f$	Δ^{s-1}	$2^s \binom{f}{s}$	Δ^{s-1}	$2^s \binom{f}{s}$
$s = f + 1$	-	-	Δ^{f-1}	2^f

Table 5.10: Number of vertices of each type within submaximal levels of extremal and largest-known graphs of degrees 6 to 9 (dimension $f = 3$ and 4)

Vertex type T_s	Even degree $d = 2f$		Odd degree $d = 2f + 1$	
	Difference order	Value	Difference order	Value
$2 \leq s \leq f$	Δ^{s-1}	$-(s-1)2^s \binom{f}{s}$	Δ^{s-1}	$-(s-1)2^s \binom{f}{s}$
$s = f + 1$	Δ^{f-2}	$f2^{f-1}$	Δ^{f-1}	$-(f-1)2^f$
$s = f + 2$	-	-	Δ^{f-2}	$f2^{f-1}$

We will now prove for circulant graphs of any degree d and arbitrary diameter k that the number of vertices of each type in each maximal level is determined by the same upper bound $Upp_{AbCay}(d, k)$ that has been proved to determine the total number of vertices in each maximal level.

Theorem 5.5. *For circulant graphs of any degree d , the number of vertices, $VT(d, s, l)$, of type T_s in distance partition level $l \geq 1$, where the level is maximal, is given by the following formulae.*

For even degree $d = 2f$ where f is the dimension, we have:

$$VT(d, s, l) = \binom{f}{s} \sum_{i=1}^s (-1)^{s-i} \binom{s}{i} Upp_{AbCay}^{level}(2i, l)$$

and hence

$$VT(d, s, l) = \begin{cases} d & \text{for } s = 1 \\ 2fs \prod_{i=1}^{s-1} 2(f-i)(l-i)/(i+1)^2 & \text{for } s \geq 2. \end{cases}$$

For odd degree $d = 2f + 1$ where f is the dimension, we have the recurrence relation:

$$VT(d, s, l) = VT(2f, s, l) + VT(2f, s-1, l-1) \text{ for } s, l \geq 2$$

and hence

$$VT(d, s, l) = \begin{cases} d & \text{for } s = 1 \\ 2f(f-1)(l-1) + 2f & \text{for } s = 2 \\ 2fs \prod_{i=1}^{s-1} 2(f-i)(l-i)/(i+1)^2 \\ + 2f(s-1) \prod_{i=1}^{s-2} 2(f-i)(l-1-i)/(i+1)^2 & \text{for } s \geq 3. \end{cases}$$

Proof. First, consider a circulant graph of even degree d and diameter k , being the Cayley graph of a cyclic group with generating set $G = \{g_1, \dots, g_f\}$ where $f = d/2$ is the dimension of the graph. Then the connection set is $C = \{\pm g_1, \dots, \pm g_f\}$.

Let v be a vertex at distance $l < k$ from an arbitrary root vertex u . Suppose for a contradiction that a path of length l from u to v contains an edge generated by g_i and another edge generated by $-g_i$ for some $g_i \in G$. As the group is Abelian the path from u generated from the same set of edges after removing this pair would lead to v after a distance of only $l - 2$, contradicting the premise that v is distant l from u . Hence, if v is distant l from u then for any $g_i \in G$ no path of length l from u to v contains edges generated by both g_i and $-g_i$.

Suppose there exists a path p of length $l \geq 2$ from u to v with two of its edges generated by different generators, say c_1, c_2 where $|c_1| = g_i$ and $|c_2| = g_j$ for some i, j with $1 \leq i < j \leq f$. Then as the group is Abelian, we may reorder the edges of p to generate two distinct paths $p_1 = (x_1, \dots, x_{l-2}, c_1, c_2)$ and $p_2 = (x_1, \dots, x_{l-2}, c_2, c_1)$ from u to v . Now consider the two vertices v_1, v_2 both distant $l - 1$ from u , reached by following paths $p'_1 = (x_1, \dots, x_{l-2}, c_1)$ and $p'_2 = (x_1, \dots, x_{l-2}, c_2)$ from u . These are distinct vertices within distance partition level $l - 1$ that are adjacent to v in level l . Thus, v is connected to more than one level $l - 1$ vertex and so is not a type T_1 vertex. Therefore, for any type T_1 vertex in level $l \geq 1$ there is only one path from u of length l and each edge of the path is generated by the same element of the

connection set. Also, every vertex on this path is also a type T_1 vertex generated by the same element. Conversely, every element of the connection set generates a unique path from u passing through vertices which are all distinct type T_1 vertices while the distance partition level remains in the maximal zone by the definition of the upper bound $Upp_{AbCay}(d, l)$. Therefore, within the maximal zone the number of type T_1 vertices in each level, $VT(d, 1, l)$, will be equal to the degree d of the graph. This can also be expressed as the product of the number of such vertices for each generator, $Upp_{AbCay}^{level}(2, l) = 2$ where $Upp_{AbCay}^{level}(d, l) = Upp_{AbCay}(d, l) - Upp_{AbCay}(d, l - 1)$ as defined in an earlier section, and the number of generators, f , giving $VT(d, 1, l) = 2f = d$.

Next, consider any two generators $g_i, g_j \in G$ and all vertices in level l that can be reached from root vertex u with a path of length l comprised only of edges $\pm g_i$ and $\pm g_j$. As level l is maximal, by definition of the upper bound $Upp_{AbCay}(d, k)$ there are $Upp_{AbCay}^{level}(4, l)$ such vertices. We now restrict the vertex set to only those vertices where the path includes at least one edge $\pm g_i$ and one edge $\pm g_j$, so that $l \geq 2$. As the group is Abelian, each of these vertices will have at least one path from u with final edge $\pm g_i$ and at least one path with final edge $\pm g_j$, and clearly no paths with any other final edge. Therefore, all these vertices are of type T_2 . The vertices with paths only of edges $\pm g_i$ or only of edges $\pm g_j$ are excluded. Thus, the number of excluded vertices is $2Upp_{AbCay}^{level}(2, l)$, and so the number of T_2 vertices in level l reached by paths generated by the pair g_i, g_j is given by $Upp_{AbCay}^{level}(4, l) - 2Upp_{AbCay}^{level}(2, l) = 4(l - 1)$. As there are $f(f - 1)/2$ distinct pairs of generators, the total number of T_2 vertices in level l is given by $VT(d, 2, l) = f(f - 1)/2 \times 4(l - 1) = d(d - 2)(l - 1)/2$.

Similarly, the number of vertices of type T_3 in level $l \geq 3$ from any given triad of generators is $Upp_{AbCay}^{level}(6, l) - 3Upp_{AbCay}^{level}(4, l) + 3Upp_{AbCay}^{level}(2, l) = 4(l - 1)(l - 2)$. As there are $f(f - 1)(f - 2)/6$ distinct triads of generators, the total number of type T_3 vertices in level is $VT(d, 3, l) = d(d - 2)(d - 4)(l - 1)(l - 2)/12$. Also, the number of vertices of type T_4 in level $l \geq 4$ from any given set of four generators is $Upp_{AbCay}^{level}(8, l) - 4Upp_{AbCay}^{level}(6, l) + 6Upp_{AbCay}^{level}(4, l) - 4Upp_{AbCay}^{level}(2, l) = 8(l - 1)(l - 2)(l - 3)/3$, and so the total number of type T_4 vertices in level l is $VT(d, 4, l) = d(d - 2)(d - 4)(d - 6)(l - 1)(l - 2)(l - 3)/144$. More generally, for even degree $d = 2f$ and any $s \geq 1$,

$$VT(d, s, l) = \binom{f}{s} \sum_{i=1}^s (-1)^{s-i} \binom{s}{i} Upp_{AbCay}^{level}(2i, l).$$

This can be reformulated as: $VT(d, s, l) = ds \prod_{i=1}^{s-1} (d - 2i)(l - i)/(i + 1)^2$ for $s \geq 2$.

Now consider a circulant graph of odd degree $d = 2f + 1$ where f is the dimension, and order n . If the generating set is $G = \{g_1, \dots, g_f\}$ then the connection set will be $C = \{\pm g_1, \dots, \pm g_f, n/2\}$. As the element $n/2$ has order 2, it can only generate a path of length 1 to create one additional type T_1 vertex in level 1 but no extra vertices of type T_1 in any higher levels. Consider any level $l \geq 2$ within the maximal zone, and any vertex v in this level. It is possible to reach v by a path comprised either of edges generated by the non order 2 elements $\{\pm g_1, \dots, \pm g_f\}$ alone or else also by including a single edge generated by the order 2 element $n/2$. It is not possible to reach any given vertex v via paths of both cases as the level is within the maximal zone. So for any $s \geq 2$ the total number of vertices of type T_s in level l is the sum of the vertices reached by paths of either case. The number in the first case is simply the result just determined for a graph of even degree $d = 2f$: $VT(2f, s, l)$. For the vertices in the second case, where the path includes an edge $n/2$, as the group is Abelian we need consider only those paths where the final edge is $n/2$. As vertex v is of type T_s in level l , then the preceding vertex v' on each path must be of type T_{s-1} in level $l - 1$, where the path to v' is comprised of edges from the connection set $C = \{\pm g_1, \dots, \pm g_f\}$. Therefore, invoking the result for even degree again, the number of vertices in this case is $VT(2f, s - 1, l - 1)$. Hence, for a circulant graph of odd degree $d = 2f + 1$ we have $VT(d, s, l) = VT(2f, s, l) + VT(2f, s - 1, l - 1)$ for $s, l \geq 2$. \square

We have seen how an analysis of their distance partitions reveals much interesting structure of extremal and largest-known circulant graphs up to degree 9. These graphs were all found to have odd girth that is maximum for their diameter. The maximum number of vertices in each level of the distance partition was shown to be related to an established upper bound for the order of Abelian Cayley graphs, $Upp_{AbCay}(d, k)$. These graphs all have a maximal zone where the levels achieve this upper bound, and for degree $d \geq 5$ a submaximal zone where they are smaller. Defining the type of each vertex in a level according to the number of adjacent vertices in the preceding level, the number of vertices of each type in each maximal level was also shown to be related to the same upper bound. Finally, the total number of type T_1 vertices in each of these graphs was determined to be a linear function of their diameter.

We have observed for all the extremal and largest-known graphs of degree 4 to 9 that the total number of type T_1 vertices increases by 4 for every increase by 1 in the diameter. We have also established that the number of type T_1 vertices in each level $l \geq 2$ within the maximal zone is twice the dimension f , giving $2f$. The resultant ratio of $2/f$ gives a value of 1 for degree 4, $2/3$ for degree 6, and $1/2$ for degree 8. This correlates with the proportion of levels that lie within the maximal zone for each even degree. We also note, for the largest-known graphs of degree 6 to 9, having a

submaximal zone, that the number of type T_2 vertices in each level is initially 0 in level 1, increases by $4\binom{f}{2} = 2f(f-1)$ per level in the maximal zone, and then decreases at the same rate in the submaximal zone after a limited transition adjustment between the two zones.

5.3 Quasimaximality and maximum odd girth

Any Abelian Cayley graph of degree $d \geq 3$ has at least two distinct generators that, taken in either order, generate two distinct paths of length 2 between a single pair of vertices. Hence, these graphs have girth of at most 4. However, the odd girth of a non-bipartite Abelian Cayley graph of diameter k can vary from a minimum of 3 up to a maximum of $2k+1$. It is observed that the extremal and largest-known circulant graph families of degrees 2 to 11 all have odd girth that is maximum for their diameter. On the other hand, the circulant graphs corresponding to Chen and Jia's lower bound have lower odd girth: for example, odd girth k for degree 8 where $k \equiv 1 \pmod{4}$, and $(4k+1)/5$ for degree 10 where $k \equiv 1 \pmod{5}$.

We have already noted that all largest-known circulant graph families are quasimaximal, meaning that the first two coefficients of their order polynomials are equal to those of the Extreme Order Conjecture 3.1. For given dimension f and arbitrary diameter k , this means that the order is given by:

$$n(d, k) = \begin{cases} \frac{1}{2} \left(\frac{4}{f}\right)^f k^f + \left(\frac{4}{f}\right)^{f-1} k^{f-1} + O(k^{f-2}) & \text{for even degree } d \\ \left(\frac{4}{f}\right)^f k^f + O(k^{f-2}) & \text{for odd degree } d. \end{cases}$$

Of the circulant graph families of any degree so far discovered, it emerges that the ones with maximum odd girth ($2k+1$) are all quasimaximal. On the other hand, no family with lower odd girth has been found to be quasimaximal. These relationships are proved in Chapter 6. Subquasimaximal families that have order polynomial unchanged in the first coefficient have second coefficient reduced by an integer multiple of $(4/f)^{f-1}$. This multiple is called the *quasimaximal defect*.

Definition 5.6. If an Abelian Cayley graph family has order polynomial in the diameter k with first coefficient equal to the Extremal Order Conjecture 3.1 and lower second coefficient, then the *quasimaximal defect* of the family is the difference in the second coefficient expressed as a multiple of $(4/f)^{f-1}$.

Some examples for degree 7, diameter class $0 \pmod{3}$, are shown in Table 5.11. The range of valid odd-girth defect increases with increasing quasimaximal defect.

Table 5.11: The order and odd girth of some degree 7 circulant graph families of diameter class $k \equiv 0$ with increasing quasimaximal defect

Quasimaximal defect (second order coeff)	Order polynomial in $2a$ ($a = 2k/3$)	Odd girth	Odd-girth defect
Quasimaximal		Maximum	
0	(1 0 3 0)	$2k + 1$	0
Subquasimaximal		Lower	
1	(1 -1 0 0)	$2k - 1$	2
	(1 -1 4 0)	$4k/3 + 1$	$2k/3$
2	(1 -2 1 0)	$2k - 1$	2
	(1 -2 7 0)	$2k - 3$	4
	(1 -2 6 -6)	$4k/3 + 3$	$2k/3 - 2$
	(1 -2 4 -2)	$4k/3 + 1$	$2k/3$
	(1 -2 3 -2)	$4k/3 - 1$	$2k/3 + 2$
3	(1 -3 9 -4)	$2k - 1$	2
	(1 -3 15 0)	$2k - 3$	4
	(1 -3 5 0)	$2k - 5$	6
	(1 -3 12 0)	$4k/3 + 3$	$2k/3 - 2$
	(1 -3 10 0)	$4k/3 + 1$	$2k/3$
	(1 -3 2 0)	$4k/3 - 1$	$2k/3 + 2$
4	(1 -4 11 -12)	$2k - 1$	2
	(1 -4 9 0)	$2k - 3$	4
	(1 -4 21 0)	$2k - 5$	6
	(1 -4 7 -24)	$2k - 7$	8
	(1 -4 9 14)	$4k/3 + 5$	$2k/3 - 4$
	(1 -4 8 -20)	$4k/3 + 3$	$2k/3 - 2$
	(1 -4 14 0)	$4k/3 + 1$	$2k/3$
	(1 -4 3 0)	$4k/3 - 1$	$2k/3 + 2$

Furthermore, for all known circulant and Abelian Cayley graph families of arbitrary dimension f , applying the standard conversion from diameter k to parameter $2a$ where $a = (2k + c)/f$ for corresponding constant c , such that the order polynomial $n(2a)$ is given by:

$$n(2a) = \begin{cases} (e_f \dots e_0)/2 & \text{for even degree} \\ (e_f \dots e_0) & \text{for odd degree,} \end{cases}$$

then the e_i are all integral. These observed relationships are proved for all Abelian Cayley graphs in Chapter 6.

5.4 Conjugation

Two interesting relationships, conjugation and translation, may be observed between the families of largest-known degree 6 and 7 circulant graphs and also between those of degree 10 and 11. These relationships are not apparent when the formulae are

expressed in conventional terms as polynomials in the diameter k , but become evident when expressed in terms of the parameter $a = (2k + c)/f$, with constant c .

The first of these relationships, conjugation, relates quasimaximal graph families of the same degree. The formulae for the order of some pairs of graph families have an alternating inverse relation, where the coefficients of one are alternately equal and negative to the other. Moreover, each generating set of one family is similarly related to a generating set of the other family. This conjugate relationship may be seen for the order of largest-known degree 6 graph families between diameter classes $k \equiv 0$ and $k \equiv 2 \pmod{3}$ in Tables 2.3 and 4.1, and also for the order and generating sets of degree 10 graph families, between F10:0 and F10:4 and between F10:1 and F10:3 in Tables 4.6 and 4.8. For ease of reference, the example for F10:0 and F10:4 is represented here in Table 5.12.

Table 5.12: Order and generating sets for degree 10 graph families F10:0 and F10:4, demonstrating the conjugation

Order and generating set	Family F10:0, $k \equiv 0 \pmod{5}$ Polynomial in $2a$ where $a = 2k/5$	Family F10:4, $k \equiv 4 \pmod{5}$ Polynomial in $2a$ where $a = (2k + 2)/5$
$LK_{circ}(10, k)$	(1 2 8 8 5 2) /2	(1 -2 8 -8 5 -2) /2
g_1	(0 0 1 3 2 0) /2	(0 0 1 -3 2 0) /2
g_2	(0 0 2 0 0 0) /2	(0 0 2 0 0 0) /2
g_3	(0 0 2 3 3 2) /2	(0 0 2 -3 3 -2) /2
g_4	(0 1 2 4 5 2) /2	(0 1 -2 4 -5 2) /2
g_5	(0 1 1 2 0 0) /2	(0 1 -1 2 0 0) /2

The formulae defining both graph families (order and generating set) are essentially the same, with the only difference being that the parameter a is taken positive in one case and negative in the other. A worked example of this is presented in Table 5.13 where the order of F10:4 is defined by the formulae for F10:0 using negative values for a . The same logic applies for their generators.

Table 5.13: Example of two degree 10 graph families sharing a common definition of their order

Family F10:4	Common order, $n = (1 2 8 8 5 2) / 2$							Family F10:0
$k \equiv 4 \pmod{5}$	$k :$	9	4	-	0	5	10	$k \equiv 0 \pmod{5}$
$-a = (2k + 2)/5$	$a :$	-4	-2	-	0	2	4	$a = 2k/5$
-Order	$n :$	-14099	-457	-	1	1099	22805	Order

We have seen that the graph family F10:4 is conjugate to F10:0, and that F10:3 is conjugate to F10:1. It is easily seen that F10:2 is self-conjugate. More generally, by definition of the method of construction, for any quasimaximal circulant graph family defined by formulae with positive values of a , taking negative values of a will produce a valid quasimaximal graph family. Conjugation only applies to quasimaximal graph

families because of the inversion of the second coefficient of the order polynomial. Thus, the conjugate of a graph family that is subquasimaximal in the second coefficient would have a second coefficient higher than quasimaximal, contrary to the Extremal Order Conjecture 3.1. It can readily be checked that for degree 6, the families of largest-known circulant graphs for the classes $k \equiv 0$ and $k \equiv 2 \pmod{3}$ are also conjugate pairs, with the family for $k \equiv 1$ being self-conjugate. This is also true for degree 8, between the families of largest-known graphs for the two classes $k \equiv 0$ and $k \equiv 1 \pmod{2}$ [23]. The alternating inverse relation for the orders of these families is shown in Table 5.14.

Table 5.14: Order of largest-known degree 6 and 8 circulant graphs, demonstrating the conjugation

Degree d	Diameter $k \pmod{f}$	Order, $LK_{circ}(d, k)$ in terms of k	Order, $LK_{circ}(d, k)$ in terms of $2a$	where $a =$
6	0	$(32k^3 + 48k^2 + 54k + 27)/27$	$(1 \ 2 \ 3 \ 2) / 2$	$2k/3$
	1	$(32k^3 + 48k^2 + 78k + 31)/27$	$(1 \ 0 \ 3 \ 0) / 2$	$(2k + 1)/3$
	2	$(32k^3 + 48k^2 + 54k + 11)/27$	$(1 \ -2 \ 3 \ -2) / 2$	$(2k + 2)/3$
8	0 & 2	$(k^4 + 2k^3 + 6k^2 + 4k)/2$	$(1 \ 2 \ 6 \ 4 \ 0) / 2$	$k/2$
	1 & 3	$(k^4 + 2k^3 + 6k^2 + 6k + 1)/2$	$(1 \ -2 \ 6 \ -4 \ 0) / 2$	$(k + 1)/2$

The same conjugation applies equally between pairs of quasimaximal graph families of odd degree. For any dimension f and for any k with $0 \leq k < f$, we find that a family of degree d and diameter class k is the conjugate of a family with diameter class $k^* = f - 1 - k$ if d is even and with diameter class $k^* = f - k$ if d is odd. This is captured in Conjecture 5.7.

Conjecture 5.7. *Let $\mathcal{X} = \{X(d, k) : k \in K\}$ be a quasimaximal circulant graph family of degree d and corresponding dimension $f = \lfloor d/2 \rfloor$ for some diameter class K with root diameter $k_{\mathcal{X}}$ so that $0 \leq k_{\mathcal{X}} < f$. Let its order be a polynomial $n_{\mathcal{X}}(2a)$ of degree f , with generating set $\{g_1(2a), g_2(2a), \dots, g_f(2a)\}$ with $a = (2k + c)/f$ where $c = (2(f - k_{\mathcal{X}}) + \lfloor f/2 \rfloor) \pmod{f - \lfloor f/2 \rfloor}$*

Then there exists a quasimaximal circulant graph family $\mathcal{Y} = \{Y(d, k') : k' \in K_{\mathcal{Y}}\}$, conjugate to \mathcal{X} , for diameter class $K_{\mathcal{Y}}$ with root diameter $k_{\mathcal{Y}}$, with order $n_{\mathcal{Y}}(2a') = n_{\mathcal{X}}(-2a')$ and generating set $\{g_1(-2a'), \dots, g_f(-2a')\}$, where

$$\begin{cases} k_{\mathcal{Y}} = f - 1 - k_{\mathcal{X}} \text{ and } a' = (2k' + 2 - c)/f & \text{for even } d \\ k_{\mathcal{Y}} = f - k_{\mathcal{X}} \text{ and } a' = (2k' - c)/f & \text{for odd } d. \end{cases}$$

5.5 Translation

The second relationship, translation, is only observed between pairs of Abelian Cayley graph families of the same odd dimension. For circulant graph families, this is a given family of odd order and even degree and another of odd degree. This relation is only reflected in the formulae for the order and generators when expressed as polynomials in the parameter $2a$ rather than the diameter k . For dimensions $f = 3$ and 5 , with appropriate pairing of diameter $k \pmod{f}$ and definition of the parameter a in terms of k , the polynomial in a for the order of each family of largest-known odd-order degree $2f$ circulant graphs is exactly half the corresponding polynomial for the order of the family of largest known degree $2f + 1$ graphs. The generating set for the degree $2f$ family is the same as the one for the degree $2f + 1$ family, modulo the order of the graph. This relationship is demonstrated for the polynomials of graph order in Table 5.15, and presented below as Theorem 5.8.

Table 5.15: Correspondence between the order of families of largest known circulant graphs of degree $2f$ with odd order and degree $2f + 1$, for odd dimension $f = 3$ and 5

Diam k (mod 3)	Largest known odd-order degree 6 $LK_{circ}(6, k)$	where $a =$	Diam k (mod 3)	Largest known degree 7 $LK_{circ}(7, k)$	where $a =$
0	(1 2 3 2)/2	$2k/3$	2	(1 2 3 2)	$(2k - 1)/3$
1	(1 0 3 0)/2	$(2k + 1)/3$	0	(1 0 3 0)	$2k/3$
2	(1 -2 3 -2)/2	$(2k + 2)/3$	1	(1 -2 3 -2)	$(2k + 1)/3$
Diam k (mod 5)	Largest known odd-order degree 10 where $a =$		Diam k (mod 5)	Largest known degree 11 $LK_{circ}(11, k)$	where $a =$
0	(1 2 8 8 5 2)/2	$2k/5$	3	(1 2 8 8 5 2)	$(2k - 1)/5$
1	(1 -4 12 -16 9 -4)/2	$(2k + 3)/5$	4	(1 -4 12 -16 9 -4)	$(2k + 2)/5$
2	(1 0 6 0 5 0)/2	$(2k + 1)/5$	0	(1 0 6 0 5 0)	$2k/5$
3	(1 4 12 20 15 4)/2	$(2k - 1)/5$	1	(1 4 12 20 15 4)	$(2k - 2)/5$
4	(1 -2 8 -8 5 -2)/2	$(2k + 2)/5$	2	(1 -2 8 -8 5 -2)	$(2k + 1)/5$

$$(c_5 c_4 c_3 c_2 c_1 c_0)/b = (c_5(2a)^5 + c_4(2a)^4 + c_3(2a)^3 + c_2(2a)^2 + c_1(2a) + c_0)/b$$

Theorem 5.8. For any odd dimension f and any $k_{\mathcal{X}}$ where $0 \leq k_{\mathcal{X}} < f$, let $\mathcal{X}(2f + 1, k_{\mathcal{X}})$ be a family of Abelian Cayley graphs $X(2f + 1, k)$ of odd degree $2f + 1$ and diameter k for any $k \equiv k_{\mathcal{X}} \pmod{f}$, with order defined by a polynomial $n_{\mathcal{X}}(2a)$ of degree f in the parameter $a = (2k + c)/f$ where $c \in \{-(f - 1)/2, \dots, (f - 1)/2\}$ such that $c \equiv -2k_{\mathcal{X}} \pmod{f}$, c being chosen so that a is integral, and with generating set $\{g_1(2a), \dots, g_f(2a)\}$ where $g_i(2a)$ are polynomials of degree at most f and taken mod $n_{\mathcal{X}}(2a)$.

For $k_{\mathcal{Y}} = (k_{\mathcal{X}} + (f - 1)/2) \pmod{f}$, we define $\mathcal{Y}(2f, k_{\mathcal{Y}})$ to be the family of Abelian Cayley graphs $Y(2f, k')$ of even degree $2f$ for any $k' \equiv k_{\mathcal{Y}} \pmod{f}$ with order

$n_Y(2a') = n_X(2a')/2$ where $a' = (2k' + c + 1)/f$ and with generating set $\{g_1(2a'), \dots, g_f(2a')\} \pmod{n_Y(2a')}$.

Then for any such k' , $Y(2f, k')$ has diameter k' .

Proof. Let the lattice generating vectors for X be $\mathbf{v}_1, \dots, \mathbf{v}_f$, and the corresponding lattice L_X . Let the involutory vector $\mathbf{v}_m = \frac{1}{2} \sum_{i=1}^f \mathbf{v}_i$. Then by Proposition 2.2, $(S_{f,k} + L_X) \cup (S_{f,k-1} + \mathbf{v}_m + L_X) = \mathbb{Z}^f$.

We can picture $S_{f,k} + L_X$ as forming an f -dimensional ‘chess board’ of black squares covered by Lee spheres of radius k centred on the lattice points of L_X . The spaces in between, the white squares, are covered by $S_{f,k-1} + \mathbf{v}_m + L_X$, Lee spheres of radius $k - 1$ centred on a copy of the lattice translated by \mathbf{v}_m . As the union $(S_{f,k} + L_X) \cup (S_{f,k-1} + \mathbf{v}_m + L_X)$ is a covering of \mathbb{Z}^f , any point P of \mathbb{Z}^f either lies within distance k of the nearest lattice point Q_1 of L_X or lies within distance $k - 1$ of the nearest lattice point Q_2 of $\mathbf{v}_m + L_X$, or possibly both if P lies within an intersection of the two neighbouring Lee spheres. Therefore, the distance between Q_1 and Q_2 , by the Manhattan norm, $\delta(Q_1, Q_2) \leq 2k$.

Let L_Y be the lattice corresponding to graph Y . By construction, $\mathbf{v}_m \in L_Y$ and hence $L_Y = L_X \cup (\mathbf{v}_m + L_X)$. In this case, all the Lee spheres have radius k' , the diameter of Y . So the point P is within a distance k' of one or other of Q_1 and Q_2 , or both. Hence $\delta(Q_1, Q_2) \leq 2k' + 1$.

First, consider the case $k_X = 0$ for family \mathcal{X} . Given any $k \equiv k_X \pmod{f}$, let X be the graph in \mathcal{X} with diameter k . Then, applying the substitution $a = 2k/f$, we have $\delta(Q_1, Q_2) \leq 2k = af$. As the lattice points for family \mathcal{Y} are defined by the same vectors as for the combined set of lattice points for family \mathcal{Y} , we must also have $2k' + 1 = af$, so that $a = (2k' + 1)/f$. This substitution gives integer values for a if and only if $k' \equiv (f - 1)/2 \pmod{f}$ and f is odd. Similarly, for any diameter class $k_X < f$ for \mathcal{X} , the corresponding diameter class for \mathcal{Y} is $k_Y = (k_X + (f - 1)/2) \pmod{f}$, again only in case f is odd. \square

Whilst for quasimaximal circulant graph families, translation is only evident when the order of the even-degree family is odd, it emerges that translation actually occurs between all Abelian Cayley graph families of odd dimension. For even-order even-degree quasimaximal circulant graph families, their translates are Abelian Cayley graph families of cyclic rank 2, and therefore are not apparent as a relation between circulant graph families. The relationship describing all cases investigated depends on whether the even-degree family has at least one odd cyclic order or if they are all

even. If the even-degree Abelian Cayley graph family has at least one odd cyclic order, then its odd-degree translate has equal cyclic rank. However, if the even-degree family has only even cyclic orders, then its translate has cyclic rank increased by 1 with a corresponding cyclic order that has constant value. This rule applies equally to all quasimaximal circulant graph families investigated. It is summarised in Table 5.16.

Table 5.16: Translation between pairs of quasimaximal Abelian Cayley graph families of odd dimension f

Even degree $d_e = 2f$		Odd degree $d_o = 2f + 1$	
Cyclic rank	At least one odd cyclic order	Cyclic rank	Smallest cyclic order
r	yes	r	-
r	no	$r+1$	constant

It appears that translation causes the number of even cyclic orders to be increased by 1. If the even-degree family contains odd cyclic orders, then one of these becomes even, otherwise an extra constant-valued even cyclic order is created, increasing by 1 the cyclic rank of the family. This has not been proved and remains a conjecture.

Conjecture 5.9. *For odd dimension f , let \mathcal{G} be a quasimaximal Abelian Cayley graph family of even degree $2f$, and let \mathcal{H} be its translate family. Suppose \mathcal{G} has cyclic rank r , of which s have odd cyclic order and the others even.*

If $s \geq 1$, then \mathcal{H} also has cyclic rank r . The number with odd cyclic order is reduced by 1 to $s - 1$, and the number with even cyclic order increased by 1 to $r - s + 1$.

If $s = 0$, then \mathcal{H} has cyclic rank increased by 1 to $r + 1$, all of even cyclic order. Moreover, the smallest cyclic order is constant for all diameters.

Some examples for dimension 7 are presented in Table 5.17.

This relationship does not hold for subquasimaximal families. Table 5.18 shows an example of an even-order subquasimaximal degree 6 circulant graph family translating to an Abelian Cayley graph family of cyclic order 2, consistent with the quasimaximal relationship. It also shows two examples where the graph families translate to circulant graph families instead.

Table 5.17: Translation examples for quasimaximal Abelian Cayley graph families of dimension 7

Degree 14 Abelian Cayley graph families								Degree 15 Abelian Cayley graph families												
Diam				Cyclic orders				Diam				Cyclic orders								
class								class												
Odd-order degree 14 circulant graph families																				
0	(1	2	14	20	27	18	11	2)	/2	4	(1	2	14	20	27	18	11	2)		
1	(1	6	28	76	127	126	67	14)	/2	5	(1	6	28	76	127	126	67	14)		
3	(1	0	14	0	21	0	7	0)	/2	0	(1	0	14	0	21	0	7	0)		
								odd										even		
Even-order degree 14 circulant graph families																				
0	(1	2	15	20	21	12	4	0)	/2	4	(1	2	15	20	21	12	4	0)	/4 × (4)	
2	(1	-4	21	-46	50	-30	8	0)	/2	6	(1	-4	21	-46	50	-30	8	0)	/8 × (8)	
								even										even	even	
Degree 14 Abelian Cayley graph families of cyclic rank 2 with at least one odd cyclic order																				
1	(1	5	24	56	69	43	10)	/2 × (1	1)	5	(1	5	24	56	69	43	10)	× (1	1)	
								odd										even	odd	
3	(1	0	13	-2	7	2	0)	× (1	0)/2	0	(1	0	13	-2	7	2	0)	× (1	0)	
								even										even	even	
Degree 14 Abelian Cayley graph families of cyclic rank 2 with all cyclic orders even																				
0	(1	2	16	20	13	6	0)	/2 × (1	0)	4	(1	2	16	20	13	6	0)	/2 × (1	0)	× (2)
1	(1	5	25	59	54	16	0)/8	× (4	4)	5	(1	5	25	59	54	16	0)	/4 × (2	2)	× (2)
2	(1	-3	19	-29	12	0)	/2 × (1	-1	0)	6	(1	-3	19	-29	12	0	0)/16	× (4	-4)	× (4)
								even										even	even	even

Table 5.18: Translation examples for even-order subquasimaximal circulant graph families of degree 6

LGM (odd basis)	Degree 6 circulant families		Degree 7 Abelian Cayley families	
	Diameter class	Order	Diameter class	Cyclic orders
Translating to degree 7 Abelian Cayley graph family of cyclic rank 2				
$\begin{pmatrix} 2a & -1 & -1 \\ 0 & 2a & 0 \\ 2 & 1 & 2a+1 \end{pmatrix}$	0	$(1\ 1\ 2\ 0)/2$ even	2	$(1\ 1\ 2\ 0)/2 \times (2)$ even even
Translating to degree 7 circulant graph family				
$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & 0 \\ 1 & 1 & 2a+1 \end{pmatrix}$	0	$(1\ 1\ 2\ 0)/2$ even	2	$(1\ 1\ 2\ 0)$ even
$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -2 \\ 1 & 1 & 2a-1 \end{pmatrix}$	1	$(1\ -1\ 4\ 0)/2$ even	0	$(1\ -1\ 4\ 0)$ even

THE LATTICE GENERATOR MATRIX OF AN ABELIAN CAYLEY GRAPH FAMILY

In Chapter 6, we define the canonical lattice generator matrix (LGM) of an Abelian Cayley graph family. Some interesting properties and relations are discussed for both quasimaximal and subquasimaximal graph families: radius maximality and eccentricity. An important theorem is established that proves the existence of all graphs in an Abelian Cayley graph family given the existence of graphs of low diameter. The equivalence is established of a graph family being quasimaximal, its graphs having maximum odd girth and its canonical LGM being radius maximal. In the final section, the graph family relationships of translation, conjugation and transposition are defined in terms of their canonical LGMs.

The first two sections of this chapter define canonical formats for lattice generator matrices of even and odd degree. They are defined in a natural way. For odd degree, where each vector has one distinct element that includes the parameter a , the vectors are ordered so that these elements lie on the leading diagonal. For even degree, where each element of each vector includes this parameter, the vectors are chosen and ordered in a way that is simply derived from the canonical odd-degree format, as will be made evident in Section 6.3.

6.1 Canonical even-degree lattice generator matrices

The lattice generator matrices constructed by Dougherty and Faber in their existence proof of the largest-known degree 6 circulant graph families for the three diameter classes display certain common features and may be presented in a standard, canonical format. As mentioned in Section 2.1, lattice generator matrices for even-degree graph families of dimension f are composed of any f independent vectors, \mathbf{v}_i for $1 \leq i \leq f$, out of a set of 2^{f-1} along with their inverses (2^f in total). The vector elements all have format $\pm(a \pm c_{ij})$ where c_i are constants and $a = (2k + c)/f$ for appropriate constant c . Therefore, the lattice vectors each have Manhattan length $fa = 2k$ plus a constant (or $\sqrt{f}a$ in Euclidean norm). We now give the formal definition of canonical even-degree LGM format.

Definition 6.1. An $f \times f$ matrix over \mathbb{Z} is in *canonical even-degree LGM format* if all elements have format $\pm a + c_{ij}$, the coefficient of a is $+1$ in the first column and leading diagonal and -1 elsewhere, and the Manhattan length of each row vector is less than or equal to $2k + 1$ where k is the diameter of the associated graph. This restricts the choice of vectors and the order of the rows and columns to an extent.

The lattice generator matrices defined by Dougherty and Faber are shown below for the three diameter classes $k \pmod{3}$:

$$\begin{matrix} k \equiv 0 \\ \begin{pmatrix} a+1 & a & a \\ a & -a & a+1 \\ a+1 & a-1 & -a-1 \end{pmatrix} \end{matrix} \begin{matrix} k \equiv 1 \\ \begin{pmatrix} a & a & a \\ a+1 & -a & a-1 \\ a-1 & a+1 & -a \end{pmatrix} \end{matrix} \begin{matrix} k \equiv 2 \\ \begin{pmatrix} a & a & a-1 \\ a-1 & -a & a \\ a & a-1 & -a \end{pmatrix} \end{matrix}.$$

These may be transformed into canonical format as follows. For $k \equiv 0$ and 2 , define a new first row by subtracting the first from the sum of the second and third, and switch the last two rows. For $k \equiv 1$, reverse the sign of the second and third columns. Then we have the following canonical set of lattice generator matrices for the degree 6 families:

$$\begin{matrix} k \equiv 0 \\ a = 2k/3 \\ \begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a & -a & a+1 \end{pmatrix} \end{matrix} \begin{matrix} k \equiv 1 \\ a = (2k+1)/3 \\ \begin{pmatrix} a & -a & -a \\ a+1 & a & -a+1 \\ a-1 & -a-1 & a \end{pmatrix} \end{matrix} \begin{matrix} k \equiv 2 \\ a = (2k+2)/3 \\ \begin{pmatrix} a-1 & -a-1 & -a+1 \\ a & a-1 & -a \\ a-1 & -a & a \end{pmatrix} \end{matrix}.$$

Note that the sum of the absolute values of the elements in each row of the three matrices, which is also the l_1 -length of each vector, equals $3a + 1$, $3a$ and $3a - 1$ respectively, which is equal to $2k + 1$ in each case. We will see that this length, $2k + 1$, is invariant for the lattice generator matrix of any known quasimaximal even-degree Abelian Cayley graph family. Such an even-degree lattice generator matrix is called *edge-maximal*.

6.2 Canonical odd-degree lattice generator matrices

The lattice generating vectors determined by Dougherty and Faber for the largest-known degree 7 circulant graph families for the three diameter classes may also be presented in canonical format. As we have seen, in the even-degree case, any independent set of f lattice generating vectors from a set of 2^{f-1} determines a lattice generator matrix. However, the odd-degree case is quite restricted in that there are only f candidate vectors to choose from: as illustrated in Section 2.1, each vector \mathbf{v}_i has an element $2a + b_i$ in a unique position and constants in the others. Therefore,

the lattice vectors each have edge length $2a$ plus a constant, in both Manhattan and Euclidean norms. Here is the formal definition of canonical odd-degree LGM format.

Definition 6.2. An $f \times f$ matrix M over \mathbb{Z} is in *canonical odd-degree LGM format* if all elements in the leading diagonal have format $2a + b_i$, all off-diagonal elements have constant value c_{ij} , the trace of M , $\text{Tr}(M) \leq 4k$ where k is the diameter of the associated graph, and each column sum is even.

The degree 7 lattice generator matrices defined by Dougherty and Faber are shown below for each diameter class $k \pmod{3}$:

$$\begin{array}{ccc}
 k \equiv 0 & k \equiv 1 & k \equiv 2 \\
 a = 2k/3 & a = (2k+1)/3 & a = (2k-1)/3 \\
 \begin{pmatrix} 2a & 1 & -1 \\ -1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix} & \begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix} & \begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix} \\
 \mathbf{v}_m = (a & a+1 & a-1) & (a & a & a-1) & (a+1 & a & a).
 \end{array}$$

These are in canonical format, with the elements that include $2a$ on the leading diagonal.

We also define an involutory vector $\mathbf{v}_m = \sum \mathbf{v}_i/2$. In fact, odd-degree graphs have two associated lattices. Apart from the principal lattice described above, we also have an *involutory* lattice, which is a translation of the principal lattice by \mathbf{v}_m . The LGM uniquely defines both lattices.

We make four observations about these matrices. Firstly, the column totals are even, ensuring that the elements of \mathbf{v}_m are integral. Secondly, the trace of each matrix equals $6a$, $6a - 2$ and $6a + 2$ respectively, which is equal to $4k$ in each case. Thirdly, the off-diagonal elements form an antisymmetric matrix with elements valued at 0 or ± 1 only, in which case the matrix is considered to have *eccentricity* 0.

Definition 6.3. Let M be a canonical odd-degree LGM, and c_{ij} and c_{ji} any transpose pair of elements of M . The *eccentricity*, $\text{eccent}(c_{ij}, c_{ji})$, of this pair is defined to be the maximum of two values: the excess above 1 of the larger absolute value, and the absolute value of their sum. Thus $\text{eccent}(c_{ij}, c_{ji}) = \max(\max(|c_{ij}|, |c_{ji}|) - 1, |c_{ij} + c_{ji}|)$. The *eccentricity* of M is defined to be the maximum eccentricity of its transpose pair elements. So $\text{eccent}(M) = \max_{i < j} (\text{eccent}(c_{ij}, c_{ji}))$.

With this definition, a canonical odd-degree LGM is off-diagonal antisymmetric with off-diagonal elements 0, 1 and -1 , if and only if its eccentricity is 0.

The first of these three properties is true for all known odd-degree Abelian Cayley graph families. The other two are true for all known quasimaximal odd-degree Abelian Cayley graph families. A less obvious property, also shared by all known quasimaximal odd-degree Abelian Cayley graph families is that the Manhattan distance from \mathbf{v}_m to each of the lattice vectors is equal to the length of the involutory vector \mathbf{v}_m , an invariant $2k$. Moreover, the distance from \mathbf{v}_m to any vertex of the lattice unit cell defined by the lattice vectors is the same invariant. Therefore, all the vertices of the unit cell lie on the boundary of a Lee sphere of radius $2k$ centred on \mathbf{v}_m . The distances from \mathbf{v}_m to the vertices are termed *radii*, and in this case the lattice generator matrix is said to be *radius maximal*.

Definition 6.4. Let M be a canonical odd-degree LGM with lattice generating vectors $\mathbf{v}_1, \dots, \mathbf{v}_f$ and involutory vector $\mathbf{v}_m = \sum \mathbf{v}_i/2$. Then M is *radius maximal* if $\delta(\mathbf{v}_i, \mathbf{v}_m) = 2k$ for all \mathbf{v}_i , where k is the diameter of the associated graph and δ is the Manhattan norm.

6.3 Translation between even and odd degree

For odd dimension, the canonical lattice generator matrices for an odd-degree graph family and its even-degree translate are directly related, and each can be determined from the other. We demonstrate this with the example of the largest-known circulant graph families for degree 6 diameter class 0, with lattice generator matrix

$M_6 = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)^T$, and degree 7 diameter class 2, with lattice generator matrix

$M_7 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)^T$. Then

$$M_6 = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{pmatrix} = \begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a & -a & a+1 \end{pmatrix}.$$

So

$$\begin{pmatrix} \sum \mathbf{w}_i - \mathbf{w}_1 \\ \mathbf{w}_2 - \mathbf{w}_1 \\ \mathbf{w}_3 - \mathbf{w}_1 \end{pmatrix} = \begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix} = M_7 = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix}.$$

The involutory vector for M_7 , $\mathbf{v}_m = \sum \mathbf{v}_i = (a+1 \ a \ a)$. So

$$\begin{pmatrix} \mathbf{v}_1 - \mathbf{v}_m \\ \mathbf{v}_2 + \mathbf{v}_1 - \mathbf{v}_m \\ \mathbf{v}_3 + \mathbf{v}_1 - \mathbf{v}_m \end{pmatrix} = \begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a & -a & a+1 \end{pmatrix} = M_6.$$

Construction 6.5. For any translate pair of Abelian Cayley graph families of odd dimension f , the following formulae are valid for conversion between their lattice generator matrices in canonical format. Let the even-degree lattice generator matrix

be $M_{2f} = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_f)^T$ and the odd-degree one $M_{2f+1} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_f)^T$. Then we have

$$\begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \dots \\ \mathbf{w}_f \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 - \mathbf{v}_m \\ \mathbf{v}_2 + \mathbf{v}_1 - \mathbf{v}_m \\ \dots \\ \mathbf{v}_f + \mathbf{v}_1 - \mathbf{v}_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \dots \\ \mathbf{v}_f \end{pmatrix} = \begin{pmatrix} \sum \mathbf{w}_i - (f-2)\mathbf{w}_1 \\ \mathbf{w}_2 - \mathbf{w}_1 \\ \dots \\ \mathbf{w}_f - \mathbf{w}_1 \end{pmatrix}.$$

It is often more useful to represent the LGM of an even-degree graph family by the matrix translated by the formulae from its canonical lattice generator matrix into canonical odd-degree format, called the *lattice generator matrix odd basis* of the family, or *LGM odd basis* for short.

Definition 6.6. For arbitrary dimension f , let M_{2f} be the canonical even-degree lattice generator matrix of an Abelian Cayley graph family \mathcal{G} of degree $2f$. Let M_{2f+1} be the corresponding matrix defined by Construction 6.5. Then M_{2f+1} is in canonical odd-degree LGM format and is the *lattice generator matrix odd basis* of the graph family \mathcal{G} , or *LGM odd basis* for short.

A noteworthy property is observed for even-degree families of even dimension. Because the dimension is even, these graph families have no corresponding odd-degree translates. Nevertheless, in all cases evaluated, applying the matrix translation formulae to the canonical even-degree lattice generator matrix produces a matrix in canonical odd-degree format. Moreover, if the even-degree family is quasimaximal, then the translated matrix has the canonical format of a quasimaximal odd-degree family, with off-diagonal elements forming an antisymmetric matrix of 0 and ± 1 elements, that is, with eccentricity 0. This is a surprising feature, given that the corresponding odd-degree graph family cannot exist.

There is just one respect in which the odd-girth format is not satisfied: the trace corresponds to a non-integral diameter class. This is, of course, a consequence of the reason why translation is invalid for even dimension f : that the conversion between diameter classes includes a factor $(f-1)/2$, which is only integral for odd dimension. Here is an example for the largest-known circulant graph family of degree 8 and diameter class 0 (mod 2), with order polynomial in $2a(1\ 2\ 6\ 4\ 0)/2$ where $a = k$:

$$\begin{array}{cc} \text{Lattice generator matrix} & \text{LGM odd basis} \\ \begin{pmatrix} a-1 & -a-1 & -a-1 & -a \\ a & a-1 & -a-1 & -a-1 \\ a & -a-1 & a-1 & -a-1 \\ a & -a & -a & a+1 \end{pmatrix} & \begin{pmatrix} 2a+1 & -1 & -1 & -1 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & -1 \\ 1 & 1 & 1 & 2a+1 \end{pmatrix} \end{array}.$$

The LGM odd basis has the canonical format of a quasimaximal degree 9 family in every respect except the trace, $8a + 2$, whereas the trace of such an LGM is always a multiple of 4.

The canonical lattice generator matrix definition for odd degree is much tighter than for even degree, for reasons already alluded to. Firstly, the structure of the odd-degree lattice only admits f lattice vectors, where f is the dimension, which is exactly the size of the matrix. In contrast, for the even-degree lattice there are 2^{f-1} distinct lattice vectors (ignoring negatives) from which an independent set of f may be chosen for the matrix, subject to the imposed restriction on the sign of a in each position.

A second reason is specific to Abelian Cayley graph families that are quasimaximal. For all known quasimaximal odd-degree families, their LGM has eccentricity 0. This important property is discussed in Section 6.4. It greatly restricts the range of admissible candidate LGMs for quasimaximal odd-degree families.

6.4 Quasimaximal graph families and their LGMs

This section considers properties of odd-degree Abelian Cayley graph families and their associated lattices and LGMs that are related to quasimaximality. These properties are:

- a graph family is quasimaximal
- the graphs in a family have maximum odd girth
- the canonical LGM of a graph family has quasimaximal format
- the canonical LGM of a graph family is radius maximal

We first need to define what it means for an LGM to be quasimaximal.

Definition 6.7. A canonical odd-degree LGM M is defined to be *quasimaximal* if it has eccentricity 0 and its trace $\text{Tr}(M) = 4k$.

Definition 6.8. A canonical even-degree LGM is defined to be *quasimaximal* if its LGM odd basis is quasimaximal.

It is important to note that the term *quasimaximal* has two different but related meanings depending on whether it is describing a graph family or a canonical LGM. When applied to a graph family it means that the first two coefficients in the graph family's order polynomial are identical to those of the Extremal Order Conjecture 3.1. When applied to a canonical LGM it means that the LGM (LGM odd basis, for even degree) has eccentricity 0 and that its trace is equal to $4k$ where k is the diameter.

Consider an Abelian Cayley graph family of odd degree d , with corresponding dimension $f = (d - 1)/2$ and diameter class $k^* \pmod{f}$, and its canonical LGM $M = (\mathbf{v}_1 \dots \mathbf{v}_f)^T$ with involutory vector $\mathbf{v}_m = \sum \mathbf{v}_i/2$. Then the principal lattice L is generated by M , and the involutory lattice is the translate of L by the involutory vector \mathbf{v}_m . For the lattice of a graph of diameter k , an arbitrary point of \mathbb{Z}^f lies within a Manhattan distance k of a principal lattice point or distance $k - 1$ of an involutory lattice point. Hence $\delta(\mathbf{v}_i, \mathbf{v}_m) \leq 2k$ for all \mathbf{v}_i . In particular, $\|\sum \mathbf{v}_i\| \leq 4k$. The order of the graph is equal to the volume of the unit cell, given by the determinant of M , and the question arises whether it is maximised only when these inequalities are all at their limits. In such a case, we would have $\delta(\mathbf{v}_i, \mathbf{v}_m) = 2k$ for all \mathbf{v}_i and $\|\sum \mathbf{v}_i\| = 4k$, which is the definition of radius maximality, as observed for all quasimaximal families studied.

Theorem 6.9. *Let M be a canonical lattice generator matrix for an odd-degree Abelian Cayley graph family \mathcal{G} of dimension f with leading diagonal $(2a + b_1, \dots, 2a + b_f)$ where $a = (2k + c)/f$ for any diameter k in the diameter class and for constant c chosen to ensure a is integral. If M is radius maximal, then the off-diagonal elements are antisymmetric and the graph family \mathcal{G} is quasimaximal.*

Proof.

$$\text{Let } M = \begin{pmatrix} 2a + b_1 & & c_{1i} & \cdots & c_{1f} \\ & \ddots & & \ddots & \vdots \\ d_{i1} - c_{1i} & & 2a + b_i & & c_{if} \\ \vdots & \ddots & & \ddots & \\ d_{f1} - c_{1f} & \cdots & d_{fi} - c_{if} & & 2a + b_f \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_f \end{pmatrix},$$

so that $\mathbf{v}_1, \dots, \mathbf{v}_f$ are generating vectors for the principal lattice associated with the graph family \mathcal{G} . Let \mathbf{v}_m be the corresponding involutory vector, equal to half the column totals of M . Observing that all the c_{ij} terms cancel out, and with δ as the Manhattan norm, we have

$$\delta(\mathbf{0}, \mathbf{v}_m) = fa + \sum_{g=1}^f b_g/2 + \sum_{g=2}^f \sum_{h=1}^{g-1} d_{gh}/2.$$

$$\delta(\mathbf{v}_i, \mathbf{v}_m) = fa + \sum_{g=1}^f b_g/2 + \sum_{g=2}^f \sum_{h=1}^{g-1} d_{gh}/2 - \sum_{h=1}^{i-1} d_{ih} - \sum_{g=i+1}^f d_{gi}.$$

$$\begin{aligned} \delta(\mathbf{v}_i + \mathbf{v}_j, \mathbf{v}_m) &= fa + \sum_{g=1}^f b_g/2 + \sum_{g=2}^f \sum_{h=1}^{g-1} d_{gh}/2 - \sum_{h=1}^{i-1} d_{ih} - \sum_{g=i+1}^f d_{gi} \\ &\quad - \sum_{h=1}^{j-i} d_{jh} - \sum_{g=j+1}^f d_{gj} + 2d_{ij}, \text{ for } i > j. \end{aligned}$$

Note that the three above expressions all include a term $fa + \sum_{g=1}^f b_g/2$ and a term representing half the sum of the d_{gh} with each d_{gh} taken either positive or negative in distinct combinations. As M is radius maximal, these three expressions all equal $2k$

for any i, j . We have $1 + f + (f - 1)(f - 2)/2$ distinct combinations of the d_{gh} to solve for the values of the $(f - 1)(f - 2)/2$ different d_{gh} . The trivial solution that $d_{gh} = 0$ for all g, h is therefore the only solution, and so M is off-diagonal antisymmetric. Consequently, we have $fa + \sum_{g=1}^f b_g/2 = 2k$. So $\text{Tr}(M) = 2fa + \sum_{g=1}^f b_g = 4k$. Therefore, the graph family \mathcal{G} is quasimaximal. \square

However, it is not the case that every matrix in the format of a radius-maximal canonical quasimaximal odd-degree LGM has an associated Abelian Cayley graph family. Consider the case for degree 9, diameter class $k \equiv 0 \pmod{2}$ with matrix

$$L = \begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & 1 \\ 0 & 1 & -1 & 2a \end{pmatrix}, \text{ where } a = k/2.$$

L is off-diagonal antisymmetric with elements $|c_{ij}| \leq 1$. It is readily seen that $\text{Tr}(L) = 4k$, $\delta(\mathbf{v}_m) = 2k$, and $\delta(\mathbf{v}_i - \mathbf{v}_m) = 2k$ for all \mathbf{v}_i . Hence, L is in canonical quasimaximal odd-degree LGM format and is radius maximal. However, there are points within the lattice unit cell that do not lie within a distance k of any principal lattice point, nor within $k - 1$ of \mathbf{v}_m . They are the points $(1 \ a \ a \ 0)$ and $(a \ 0 \ -1 \ a)$, being the mid-points between vertices, defined by $(\mathbf{v}_2 + \mathbf{v}_3)/2$ and $(\mathbf{v}_1 + \mathbf{v}_4)/2$. Therefore, the corresponding Lee spheres do not cover \mathbb{Z}^f and so L is not the LGM of an associated Abelian Cayley graph family.

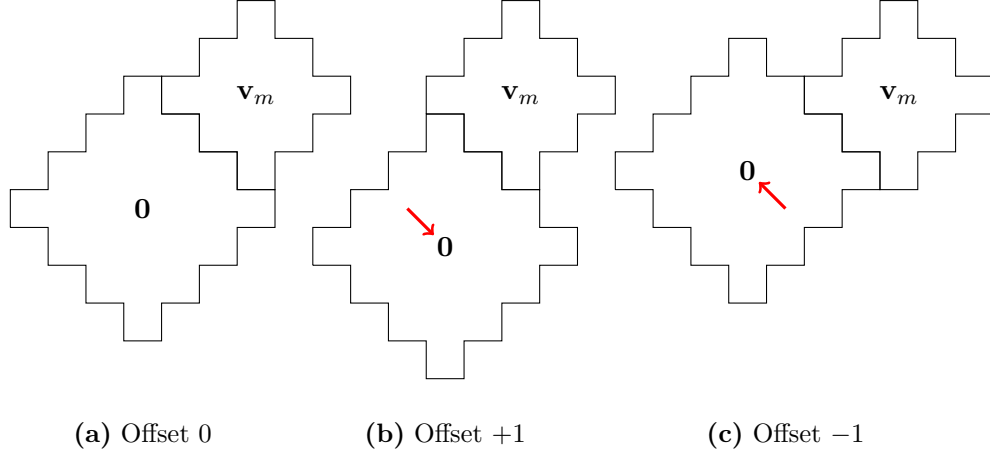
Reverting to the argument of Theorem 6.9 with its definition of the matrix M , in order for M to be in canonical quasimaximal format it remains to show that for all $i < j$, $|c_{ij}| \leq 1$. By definition, \mathbf{v}_m lies at the centre of the lattice unit cell defined by the lattice generating vectors $\mathbf{v}_1, \dots, \mathbf{v}_f$. As M is radius maximal, the vertices of the unit cell are all distant $2k$ from \mathbf{v}_m . In order for the lattice to correspond to an Abelian Cayley graph of diameter k , we require that all points \mathbb{Z}^f within the unit cell are covered by a $(k - 1)$ -sphere centred on \mathbf{v}_m and k -spheres centred on the vertices of the unit cell. By construction, the $(k - 1)$ -sphere does not intersect any of the k -spheres, instead abutting each along a common face. As defined by M , the position of the centre of each k -sphere varies across a hyperplane at constant Manhattan distance $2k$ from \mathbf{v}_m , so that the common faces can slide in any direction that maintains their non-intersecting contact.

It is instructive to explore the simplest interesting case, dimension 2 and degree 5, with diameter 3, as illustrated in Figure 2.2 of Chapter 2. For simplicity, the graph

vertex numbering is omitted from the following figures, but each square represents a point in \mathbb{Z}^2 . We will consider all configurations relative to \mathbf{v}_m and its $(k-1)$ -sphere, so that each is defined by an *offset* of two neighbouring vertices, $\mathbf{0}$ and \mathbf{v}_2 , relative to \mathbf{v}_m . In this way, any configuration is fully specified by the pair of offset values.

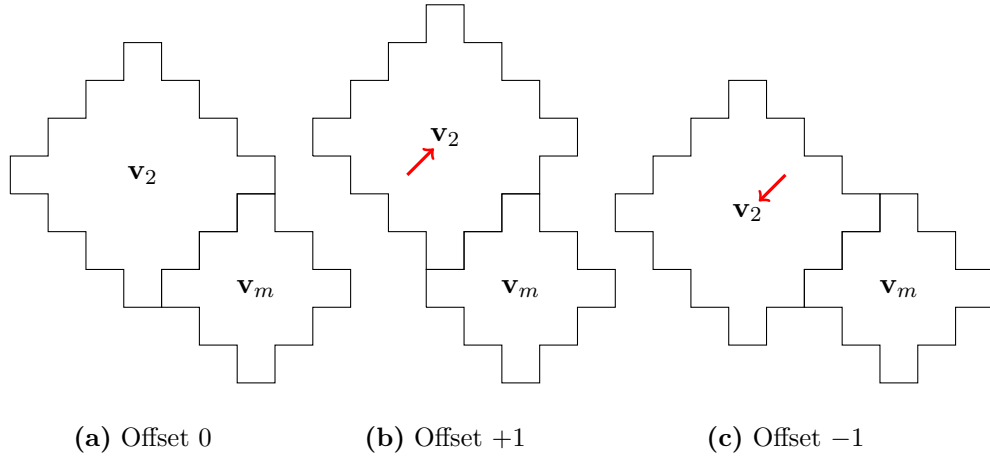
Three examples for vertex $\mathbf{0}$ are shown in Figure 6.1.

Figure 6.1: Vertex $\mathbf{0}$ with offsets 0, +1 and -1



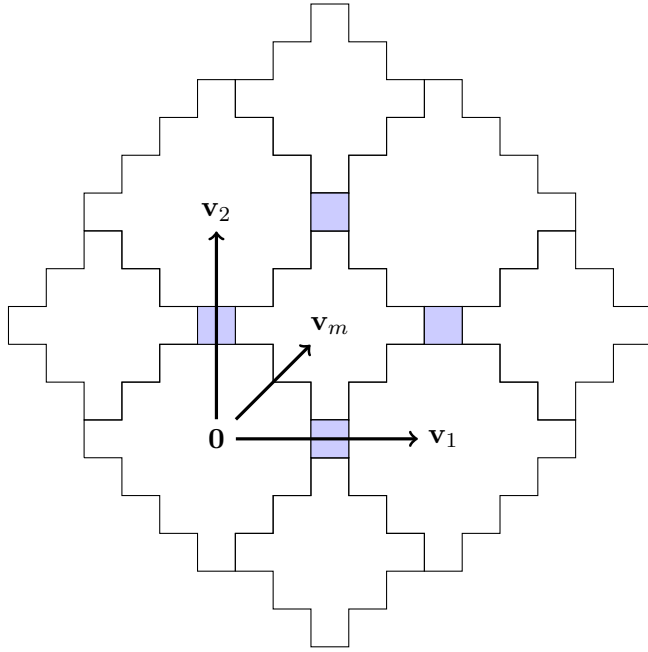
Also, three examples for vertex \mathbf{v}_2 are shown in Figure 6.2.

Figure 6.2: Vertex \mathbf{v}_2 with offsets 0, +1 and -1



In the following configurations, intersecting Lee spheres will be coloured light blue in the overlapping area. Any gap in coverage is coloured dark red. The *width* of each configuration is defined to be the difference between the two offsets.

We first consider the balanced configuration with offset $(0, 0)$, with corresponding width 0, Figure 6.3. This configuration provides full coverage of \mathbb{Z}^2 , and its associated graph family is the extremal Abelian Cayley graph family A5, of cyclic rank 2, documented in Appendix C, with order $4k^2$.

Figure 6.3: Offset (0, 0), width 0.

Offset (0, 0)

Width 0

LGM

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$$

$$\mathbf{v}_m = (3 \ 3) \begin{pmatrix} k & k \end{pmatrix}$$

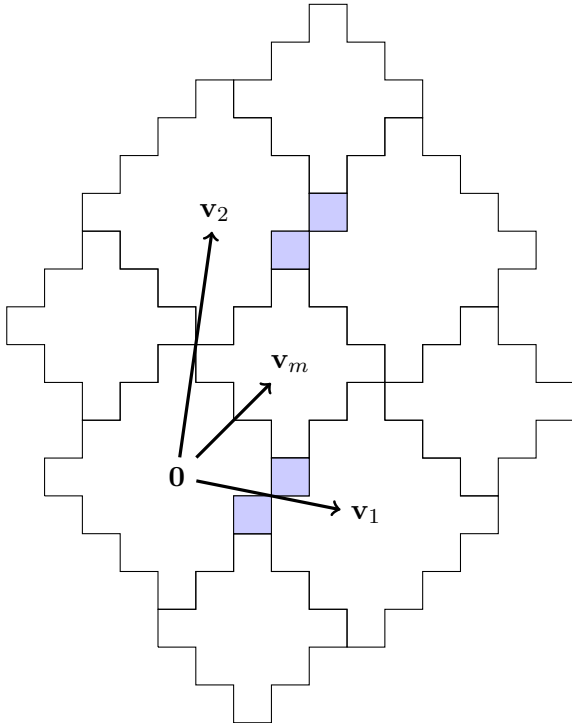
Cell size

$$2k \times 2k = 4k^2$$

Overlaps 2

Gaps 0

Next, we increase the offset of one of the vertices by 1, The alternatives are all equivalent, and this example is with offset (0, 1), and thus width 1, Figure 6.4.

Figure 6.4: Offset (0, 1), width 1.

Offset (0, 1)

Width 1

LGM

$$\begin{pmatrix} 5 & -1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 2k-1 & -1 \\ 1 & 2k+1 \end{pmatrix}$$

$$\mathbf{v}_m = (3 \ 3) \begin{pmatrix} k & k \end{pmatrix}$$

Cell size

$$(2k-1)(2k+1) + 1 = 4k^2$$

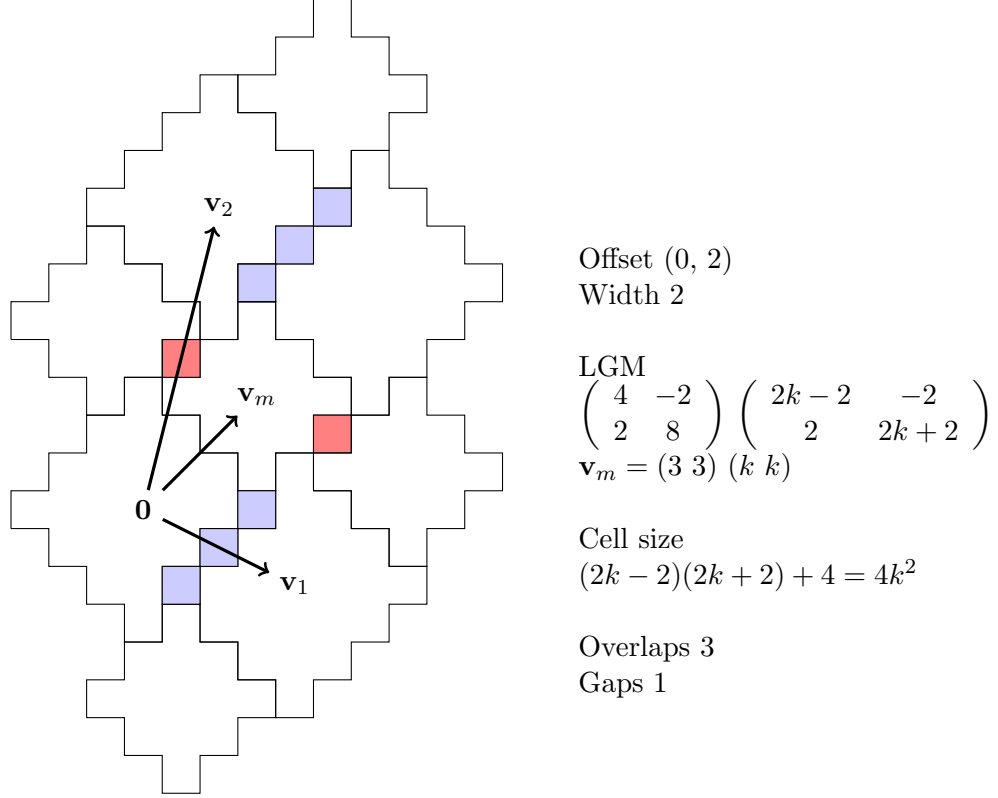
Overlaps 2

Gaps 0

This also provides full coverage of \mathbb{Z}^2 . Its graph family is the extremal circulant graph family F5 found in Appendix A, with order $4k^2$.

For the third configuration, the offset is further increased to 2, giving an offset of (0, 2) and a width of 2, Figure 6.5.

Figure 6.5: Offset (0, 2), width 2.

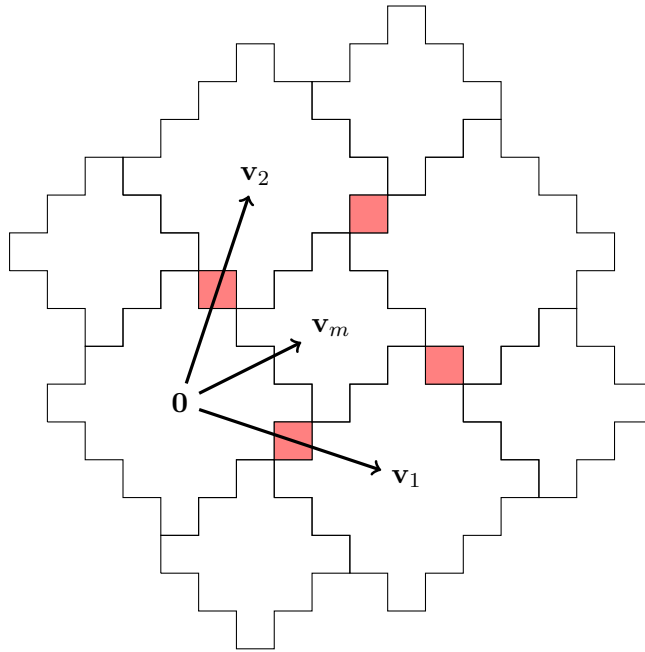


Although the unit cell size remains $4k^2$, there is an extra overlapped vertex per unit cell and also a corresponding gap. The gap represents a vertex in the associated graph that is distant more than k from reference vertex 0. Hence, this configuration does not represent a graph family.

An alternative configuration with a width 2 is obtained with an offset $(-1, 1)$, Figure 6.6.

The unit cell size is increased to $4k^2 + 4$. This exceeds the total size of the two Lee spheres by 2, which is consistent with the partial covering of \mathbb{Z}^2 with no overlaps and two gaps per unit cell.

Figure 6.6: Offset $(-1, 1)$, width 2.



Offset $(-1, 1)$
Width 2

$$\text{LGM} \begin{pmatrix} 6 & -2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 2k & -2 \\ 2 & 2k \end{pmatrix}$$

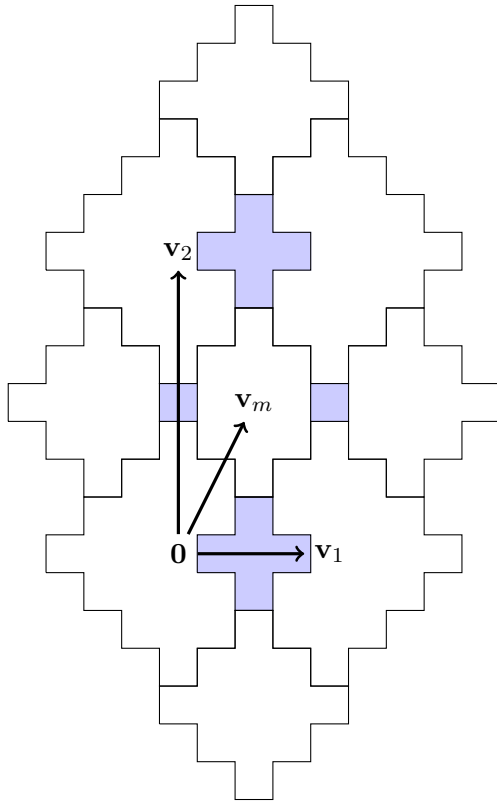
$$\mathbf{v}_m = (4 \ 2) \begin{pmatrix} k+1 & k-1 \end{pmatrix}$$

Cell size
 $4k^2 + 4$

Overlaps 0
Gaps 2

If the offset of both vertices is increased from $(0, 0)$ in step then the width remains 0.
In this final example the offset is $(1, 1)$, giving a width of 0, Figure 6.7.

Figure 6.7: Offset $(1, 1)$, width 0.



Offset $(1, 1)$
Width 0

$$\text{LGM} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 2k-2 & 0 \\ 0 & 2k+2 \end{pmatrix}$$

$$\mathbf{v}_m = (2 \ 4) \begin{pmatrix} k-1 & k+1 \end{pmatrix}$$

Cell size
 $(2k-2)(2k+2) = 4k^2 - 4$

Overlaps 6
Gaps 0

The unit cell size is reduced to $4k^2 - 4$ and is fully covered, with six overlaps. Its associated graph family is Abelian Cayley with cyclic rank 2, and although quasimaximal is clearly not extremal.

These examples illustrate some relationships which are valid for all radius-maximal degree 5 configurations.

- If the coverage is full, then the graph family exists and is quasimaximal
- If there are gaps in the coverage, then the graph family does not exist (at least not with the given diameter)
- The vector matrix in canonical odd-degree LGM format has off-diagonal elements of magnitude equal to the offset width
- If the offset width is 0, then the graph family exists and is quasimaximal Abelian Cayley with cyclic rank 2
- If the offset width is 1, then the graph family exists and is quasimaximal circulant
- If the offset width is 2 or more, then the graph family does not exist

By construction, the magnitude of the off-diagonal elements of the associated LGM is equal to the offset width, with their signs depending on the direction of the offsets. Therefore, we have quasimaximal Abelian Cayley graph families of cyclic rank 2 associated with LGMs with zero-value off-diagonal elements, quasimaximal circulant graph families with off-diagonal elements of magnitude 1, and no quasimaximal graph families with off-diagonal LGM elements of magnitude 2 or more.

All these configurations represent graph families that are quasimaximal if they exist. The basic configuration with offset $(0, 0)$ represents the fully symmetric case. It generalises directly to any dimension f for diameter class $k \equiv 0 \pmod{f}$, with offset $(0, \dots, 0)$, generating a quasimaximal Abelian Cayley graph family of maximal cyclic

$$\text{rank, } f, \text{ and LGM } \begin{pmatrix} 2a & 0 & \dots & 0 \\ 0 & 2a & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 2a \end{pmatrix} \text{ with } a = 2k/f, \text{ so that its order is } (4/f)^f k^f.$$

This establishes a simple Abelian Cayley graph lower bound for any odd degree. Increasing both offsets equally, so that the offset width is unchanged, as exemplified by comparing $(0, 0)$ with $(1, 1)$, serves only to compress the configuration along an axis, increasing the Lee sphere overlaps and decreasing the size of the unit cell.

A key question is why there is no quasimaximal degree 5 graph family for offset width 2 or above. The issue is a gap that is created near the four corners of the $(k-1)$ -sphere, where the overlaps occur with offset $(0, 0)$, Figure 6.3. When the width is increased from 0 to 1, with offset $(0, 1)$ as illustrated in Figure 6.4, but equivalently with $(0, -1)$, $(-1, 0)$ or $(1, 0)$, the overlap at that position is eliminated. Instead, the two k -spheres and two $(k-1)$ -spheres abut precisely at a point for a perfect tiling in that region. Any second step increasing the offset width to 2 opens a single gap next to this position. The two possible cases from offset $(0, 1)$ are illustrated by offsets $(0, 2)$ and $(-1, 1)$, Figures 6.5 and 6.6. To isomorphism, there are no other configurations.

We now extend the argument to higher dimension f . For a simple example, we take even dimension $f \geq 4$, diameter class $k \equiv 0 \pmod{f/2}$, odd degree $d = 2f + 1$ and

$$\text{LGM, } M = \begin{pmatrix} 2a & 0 & 0 & \dots & 0 \\ 0 & 2a & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 2a & 0 \\ 0 & 0 & \dots & 0 & 2a \end{pmatrix} \text{ where } a = 2k/f.$$

The associated lattice, L , forms a covering of \mathbb{Z}^f with Lee spheres of radius k , and the corresponding Abelian Cayley graph family is quasimaximal.

Now consider a variant of M , M' , with just one eccentric pair of elements.

$$M' = \begin{pmatrix} 2a & 2 & 0 & \dots & 0 \\ -2 & 2a & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 2a & 0 \\ 0 & 0 & \dots & 0 & 2a \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_f \end{pmatrix}.$$

It has eccentricity 1, and associated lattice L' . Its involutory vector is

$\mathbf{v}_m = (a-1 \ a+1 \ a \dots a)$. Let points \mathbf{u}_1 and \mathbf{u}_2 be defined by $\mathbf{u}_1 = (\mathbf{v}_1 + \mathbf{w})/2$ and $\mathbf{u}_2 = (\mathbf{v}_2 + \mathbf{w})/2$ where \mathbf{w} is the sum of any $(f-2)/2$ of $\mathbf{v}_3, \dots, \mathbf{v}_f$.

It is easily seen that \mathbf{u}_1 is distant $fa/2 + 1 = k + 1$ from \mathbf{v}_1 and \mathbf{v}_m and at least this distance from the other \mathbf{v}_i , and similarly for \mathbf{u}_2 . Hence L' does not form a covering of \mathbb{Z}^f with Lee spheres of radius k . Thus M' is not the LGM of an associated quasimaximal graph family. In every instance investigated of a matrix in canonical format with positive eccentricity, it was verified not to be the LGM of a quasimaximal graph family. Unfortunately, it has not yet been possible to prove that this is always the case. Therefore, this result is only presented here as a conjecture.

Conjecture 6.10. *Let M be a canonical lattice generator matrix for a quasimaximal odd-degree Abelian Cayley graph family. Then M has eccentricity 0 and is therefore quasimaximal.*

The following theorem establishes that any odd-degree Abelian Cayley graph family with radius-maximal LGM has maximum odd girth, and the converse.

Theorem 6.11. *Let \mathcal{G} be an odd-degree Abelian Cayley graph family, with lattice generator matrix M . If M is radius maximal, then the graphs of \mathcal{G} have maximum odd girth, $2k + 1$, for diameter k . In particular, the graphs are not bipartite. Conversely, if the graphs of \mathcal{G} have maximum odd girth, $2k + 1$, for diameter k , then M is radius maximal.*

Proof. Let the LGM be $M = (\mathbf{v}_1, \dots, \mathbf{v}_f)^T$ for dimension f , and let the involutory generator be \mathbf{v}_m . As M is radius maximal, the Manhattan distance from any principal lattice point of the unit cell to the centre, defined by \mathbf{v}_m , is an even distance $2k$. Thus, the distance between any two principal lattice points is also even. As the graphs are vertex transitive, we may consider any cycle to start and finish at principal lattice points in \mathbb{Z}^f . Within \mathbb{Z}^f any path that starts and finishes at the same lattice point must have even length. The only path in \mathbb{Z}^f that generates an odd length cycle in the graph is one between a principal lattice point and an involutory lattice point. This distance is even, to which a final step is added to jump from the involutory lattice point back to the principal one. The shortest path between such points has length $2k$, and therefore the shortest odd cycle has length $2k + 1$.

Conversely, suppose that M is not radius maximal. Then there is a principal lattice point, say \mathbf{v}_i , with Manhattan distance $\delta(\mathbf{v}_i, \mathbf{v}_m) < 2k$. As this distance must be even, the graph contains an odd cycle of length less than $2k + 1$. \square

6.5 Subquasimaximality and eccentricity

In this section we consider subquasimaximal Abelian Cayley graph families and their associated canonical LGMs. Definition 3.2 defined that a family is subquasimaximal if the first coefficient of its order polynomial is lower than the Extremal Order Conjecture, or if its first coefficient is equal and its second coefficient is lower.

We know that the order of a graph family is equal to the magnitude of the determinant of its lattice generator matrix. In the following, we will initially consider LGMs in canonical format for odd-degree families. Let M be the LGM in canonical format of an odd-degree graph family of dimension f , diameter class $k \equiv k^* \pmod{f}$,

with $a = (2k + c)/f$, where c is chosen to ensure a is integral, so that

$$M = \begin{pmatrix} 2a + b_1 & c_{1,2} & \dots & c_{1,f} \\ c_{2,1} & 2a + b_2 & \dots & c_{2,f} \\ \vdots & \vdots & \ddots & \vdots \\ c_{f,1} & c_{f,2} & \dots & 2a + b_f \end{pmatrix}.$$

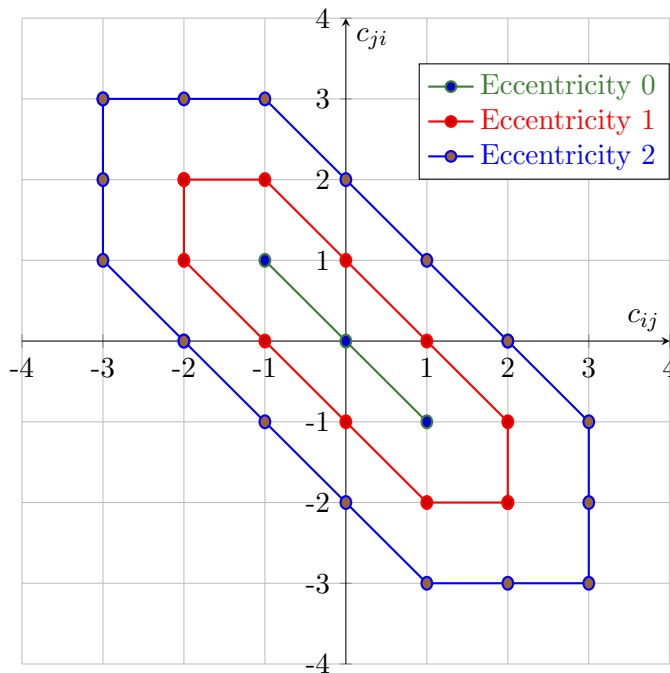
Then order

$$\begin{aligned} n = \det(M) &= (2a)^f + \sum b_i (2a)^{f-1} + \left(\sum_{i < j} b_i b_j - \sum_{i < j} c_{ij} c_{ji} \right) (2a)^{f-2} + O((2a)^{f-3}) \\ &= (1 \quad \sum b_i \quad \sum_{i < j} b_i b_j - \sum_{i < j} c_{ij} c_{ji} \quad \dots). \end{aligned}$$

Note that the second coefficient of the order polynomial in $2a$ is $\sum b_i$. If the graph family is quasimaximal, then its trace is equal to $4k$ and so $4k = 2fa + \sum b_i$. If the family is subquasimaximal, then its quasimaximal defect (see Definition 5.6) is given by $4k - 2fa - \sum b_i$.

For quasimaximal odd-degree Abelian Cayley graph families, it has been observed that their canonical LGMs have eccentricity 0, which means that their off-diagonal elements are antisymmetric with magnitude 0 or 1 only. Figure 6.8 shows the valid combinations of transpose pairs of elements for low values of eccentricity. In each case the valid region lies on or within the boundary.

Figure 6.8: Valid regions of transpose pairs for eccentricity 0 to 2



A study of over one hundred Abelian Cayley graph families of degree 7 and diameter class $k \equiv 0 \pmod{3}$, covering a range of quasimaximal defect from 0 (quasimaximal) to 4, revealed a clear relation between quasimaximal defect and eccentricity. A representative family for each defect is shown in Table 6.1. These examples all have eccentricity equal to their quasimaximal defect. However, for every quasimaximal defect investigated, graph families were found with eccentricity taking every value up to and including the defect. The sole exception is that there can be no graph families with odd quasimaximal defect and zero eccentricity because of LGM parity constraints.

Table 6.1: Quasimaximal defect and eccentricity for selected degree 7 graph families

Quasi-maximal defect	Order $a = 4k/3$	Lattice generating matrix	Max off-diag $ c_{ij} $	Max off-diag $ c_{ij} + c_{ji} $	Eccentricity
0	(1 0 3 0)	$\begin{pmatrix} 2a & -1 & 1 \\ 1 & 2a & 1 \\ -1 & -1 & 2a \end{pmatrix}$	1	0	0
1	(1 -1 5 -8)	$\begin{pmatrix} 2a-1 & -2 & -1 \\ 1 & 2a-1 & 2 \\ 2 & -1 & 2a+1 \end{pmatrix}$	2	1	1
2	(1 -2 9 -8)	$\begin{pmatrix} 2a-1 & -2 & -3 \\ 2 & 2a-1 & -1 \\ 1 & 1 & 2a \end{pmatrix}$	3	2	2
3	(1 -3 15 0)	$\begin{pmatrix} 2a-1 & -4 & -3 \\ 1 & 2a-1 & -2 \\ 2 & 1 & 2a-1 \end{pmatrix}$	4	3	3
4	(1 -4 23 0)	$\begin{pmatrix} 2a-2 & -5 & -3 \\ 1 & 2a-1 & -2 \\ 3 & 2 & 2a-1 \end{pmatrix}$	5	4	4

An extensive investigation was also undertaken on all circulant graph families of degree 6 and diameter class $k \equiv 1 \pmod{3}$ with canonical LGM M with minimum range 2, that is $M = (\pm a + b_{ij})$ where $|b_{ij}| \leq 2$. A total of 3140 families were analysed, revealing quasimaximal defect ranging up to 12 and eccentricity of their LGM odd basis not exceeding the quasimaximal defect in each case, see Table 6.2.

The reason why the eccentricity of subquasimaximal families can increase up to a maximum equal to the quasimaximal defect of the family is apparent from consideration of the corresponding lattice covering of \mathbb{Z}^f . This can most easily be seen in an example for degree 5, exploring the consequences of increasing by 1 the

Table 6.2: Number of degree 6 circulant graph families in sample, by quasimaximal defect and eccentricity

Quasi-maximal defect	Eccentricity												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	16												
1	0	115											
2	6	94	144										
3	0	46	218	98									
4	1	25	190	183	44								
5	0	16	126	187	104	28							
6	1	10	83	144	112	76	25						
7	0	11	46	93	79	83	67	14					
8	0	8	28	56	49	65	66	35	3				
9	0	7	14	23	29	36	49	32	4	0			
10	1	3	5	12	10	23	20	22	4	0	0		
11	0	1	2	2	5	8	10	8	4	0	0	0	
12	0	0	0	0	2	2	3	4	0	0	0	0	0

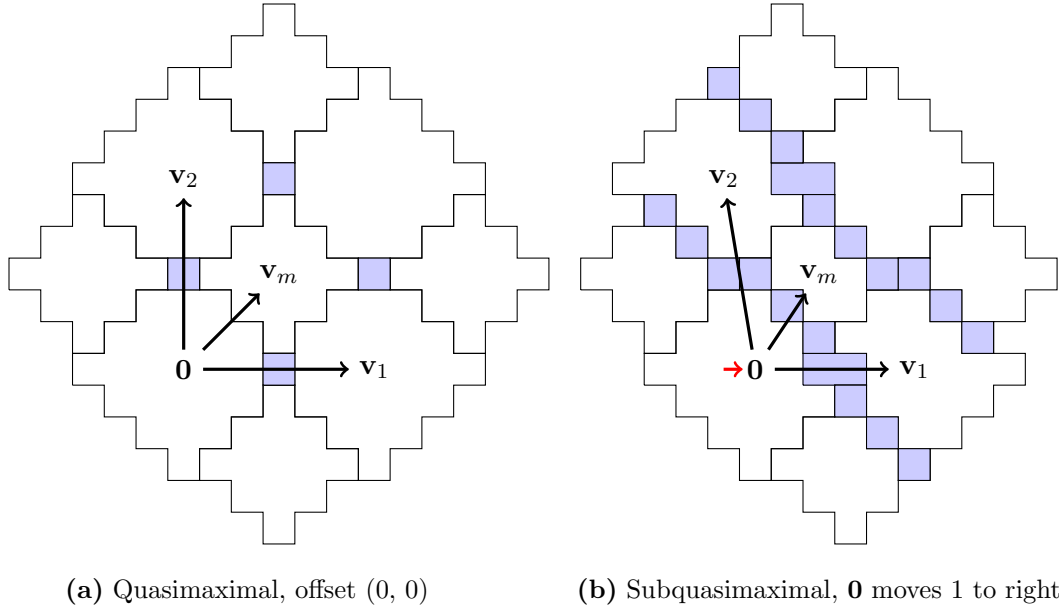
quasimaximal defect of the quasimaximal configuration with offset $(0, 0)$ presented in Section 6.4.

Relative to the centre of the lattice unit cell, \mathbf{v}_m , the quasimaximal defect is increased by 1 by moving one of the lattice points to a neighbouring point of \mathbb{Z}^2 such that its Manhattan distance from \mathbf{v}_m is reduced by 1. Starting from the configuration shown in Figure 6.9(a), quasimaximal with offset $(0, 0)$, this can be achieved in four ways:

- $(\mathbf{0}, 1, 0)$ Move $\mathbf{0}$ 1 to the right
- $(\mathbf{0}, 0, 1)$ Move $\mathbf{0}$ 1 upwards
- $(\mathbf{v}_2, 1, 0)$ Move \mathbf{v}_2 1 to the right
- $(\mathbf{v}_2, 0, -1)$ Move \mathbf{v}_2 1 downwards.

The first way, Move $(\mathbf{0}, 1, 0)$, is shown as the configuration in Figure 6.9(b).

As expected, the number of overlaps per unit cell is increased by 6 from (2 to 8), being the length of \mathbf{v}_2 in the direction normal to the move. This changes the LGM from $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ to $\begin{pmatrix} 5 & 0 \\ -1 & 6 \end{pmatrix}$. The difference is $\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$, so that the trace is reduced by 1 (necessarily equal to the increase in quasimaximal defect) and element c_{21} is also reduced by 1. Similarly, Move $(\mathbf{v}_2, 1, 0)$ changes the LGM by $\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$, reducing the trace by 1, but increasing c_{21} by 1. The other two moves change the LGM by $\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, again reducing the trace by 1 and changing element c_{12} by -1 and $+1$ respectively.

Figure 6.9: Increasing quasimaximal defect

Similarly, for every initial configuration, these four moves have the effect of reducing the trace by 1 and increasing or decreasing either off-diagonal element by 1. At least one of these changes to an off-diagonal element has the effect of increasing the eccentricity by 1. Others may leave the eccentricity unchanged or reduce it by 1, depending on the initial configuration. Now the trace defines the quasimaximal defect. So for every increase by 1 in the quasimaximal defect, the maximum attainable eccentricity is increased by 1. Therefore, for any degree 5 circulant graph family, the eccentricity will be less than or equal to the quasimaximal defect.

The restriction that a graph family with odd quasimaximal defect cannot have 0 eccentricity arises from the fact that the off-diagonal transpose pairs have even magnitude whereas the trace is odd, conflicting with the requirement that the sum of all elements of the LGM must be even.

The above arguments can be generalised to odd degree graph families and LGMs of any dimension. This is formalised in the following conjecture.

Conjecture 6.12. *For any odd-degree Abelian Cayley graph family of dimension f , the eccentricity of its lattice generator matrix is less than or equal to its quasimaximal defect. Moreover, for any quasimaximal defect, there exist graph families with every such eccentricity, with the sole exception that no graph family with odd quasimaximal defect has 0 eccentricity.*

6.6 Maximal distance partition levels of quasimaximal graphs

In Section 5.1 we discussed the distance partition profile of the extremal and largest-known circulant graph families up to degree 11. The observation was made that the proportion of maximal partition levels for any dimension f remains constant with increasing diameter at a value of $2/f$ within a small constant, as shown in Table 5.5. This proportion is valid for all quasimaximal Abelian Cayley graph families of any dimension, as demonstrated in the following theorem.

Theorem 6.13. *The number of maximal distance partition levels in a quasimaximal Abelian Cayley graph family of dimension f , for any diameter k above some small threshold k' , is equal to $(2k + s)/f$, where k' and s are constants depending on the family.*

Proof. Let M be a canonical LGM of a quasimaximal Abelian Cayley graph family of dimension f (or LGM odd basis, for even degree). Then each lattice generating vector \mathbf{v}_i has an element of magnitude $2a + b_i$ in position i , with all other elements of magnitude 0 or 1, where $a = (2k + c)/f$ for suitable constant c . Thus, the Manhattan distance between the closest pair of vertices of the lattice unit cell is $2a + e$ for some constant e with $|e| < 2(\max b_i + f)$. Consider Lee spheres of radius l centred at every unit cell vertex. None of these will intersect if $l \leq \lfloor a + e/2 \rfloor$. Therefore, distance partition level l is maximal. But if $l > \lfloor a + e/2 \rfloor$ at least two of the Lee spheres will intersect, so that the level will not be maximal. Hence, the highest maximal distance level is at $l = \lfloor a + e/2 + 1 \rfloor = \lfloor (2k + c + f(e/2 + 1))/f \rfloor$. \square

6.7 Existence proof method for graph families

The original existence proof for Abelian Cayley graph families was developed by Dougherty and Faber [10] and is described in Section 2.4. It is exemplified for degrees 8 and 10 in Sections 4.4 and 4.5, and has two parts: first identify an associated LGM, and then check all the exceptional cases that arise at the interfaces between the corresponding Lee spheres to ensure there are no gaps in the covering. For the largest-known degree 8 circulant graph families, resolution of the exceptional cases covered 16 pages [27]. For degree 10, full documentation of all exceptional cases runs to over 10,000 pages of text and was undertaken with a dedicated computer program. This is already unreadable other than by random sampling to check its validity. For higher degree, the size of the problem becomes unmanageable.

However, using knowledge of the structure of odd and even LGMs in canonical form, it is possible to prove the existence of Abelian Cayley graph families quite simply

from the existence of a single member with low diameter. The principle is that with increasing diameter, the lattice expands at the same rate as the radius of the Lee spheres centred on the lattice points, so that the boundaries of neighbouring Lee spheres maintain their relative positions where they touch or overlap. So if there is a covering of the space \mathbb{Z}^f for one diameter in its diameter class, then all higher diameters in the class will retain the covering. There is one proviso, that the sign of each element of the LGM must remain unchanged as the diameter increases, so that the distance between two neighbouring lattice points increases linearly with diameter. This defines a minimum diameter threshold for each LGM. We start with two lemmas, for even and odd degree respectively.

Lemma 6.14. *Let $K = \{k : k \equiv k^* \pmod{f}, k \geq k^*\}$ be a diameter class for some $k^* < f$. Let $a = (2k + c)/f$ where the constant c is chosen so that a is integral for any $k \in K$. Let M be an $f \times f$ matrix in canonical even-degree LGM format, and $L_k \subset \mathbb{Z}^f$ its associated lattice. If there exists a $k' \in K$ such that Lee spheres of radius k' centred on the lattice points of $L_{k'}$ form a covering of \mathbb{Z}^f , and such that the sign of each element of the LGM remains constant for all $k \in K$ with $k \geq k'$, then Lee spheres of radius k centred on the lattice points of L_k also form a covering of \mathbb{Z}^f for any $k \in K$ with $k > k'$.*

Proof. First, we consider the case with diameter k' and let $a' = (2k' + c)/f$. As M is canonical, the Manhattan length of each generating vector, \mathbf{v}_i is $a'f + e_i$ for some constants e_i . Also, the radius of the Lee spheres is $k' = (a'f - c)/2$, which is sufficient to achieve a covering of \mathbb{Z}^f .

Now the diameter is increased by f to $k' + f$, so that the parameter a is increased by 2 to $a' + 2$. Thus, the length of each lattice generating vector is increased by $2f$ and the Lee sphere radius by f . So the combined reach of the two Lee spheres centred on neighbouring lattice points is increased by $2f$, equal to the increased separation of their centres. This leaves the relative position of their common boundaries unaltered, so that the covering is maintained. \square

Lemma 6.15. *Let K and a be defined as in Lemma 6.14. Let M be an $f \times f$ matrix in canonical odd-degree format with row vector \mathbf{v}_i , involutory vector \mathbf{v}_m , and with associated principal lattice $L_k \subset \mathbb{Z}^f$ and involutory lattice L'_k . If there exists a $k' \in K$ such that Lee spheres of radius k' centred on the lattice points of $L_{k'}$ and Lee spheres of radius $k' - 1$ centred on the lattice points of $L'_{k'}$ form a covering of \mathbb{Z}^f , and such that the sign of each element of the LGM remains constant for all $k \in K$ with $k \geq k'$, then Lee spheres of radius k centred on the lattice points of L_k and Lee spheres of*

radius $k - 1$ centred on the lattice points of L'_k also form a covering of \mathbb{Z}^f for any $k \in K$ with $k > k'$.

Proof. First, we consider the case with diameter k' and let $a' = (2k' + c)/f$. As M is canonical, the Manhattan distance between any vertex \mathbf{v} of the principal unit cell of L and the point defined by \mathbf{v}_m is $a'f + e$ for some constant e depending on \mathbf{v} . Also, the sum of the radii of the Lee spheres centred on \mathbf{v} and \mathbf{v}_m is $2k' - 1 = a'f - c - 1$, which is sufficient to achieve a covering of \mathbb{Z}^f .

When the diameter is increased by f , the parameter a is increased by 2. Thus, the distance between the two lattice points is increased by $2f$ and the sum of the radii also by $2f$. Hence, the combined reach of the Lee spheres increases in line with the increased distance between their centres. This leaves the relative position of their common boundaries unaltered, so that the covering is maintained. \square

These two lemmas, 6.14 and 6.15, establish the Existence Proof Theorem for Abelian Cayley graph families.

Theorem 6.16. *Existence Proof Theorem for Abelian Cayley graph families. Let M be a matrix in the format of a canonical lattice generator matrix of either odd or even degree. If its associated Abelian Cayley graph exists for a given diameter k' within its diameter class, and such that the sign of each element of the LGM remains constant for all greater diameters within its class, then the graph family exists for all greater diameters within its class, and M is its LGM.*

Proof. Directly from Lemmas 6.14 and 6.15. \square

The existence of the covering in \mathbb{Z}^f and of the associated Abelian Cayley graph at a given diameter does not necessarily imply that each member of the family has the same cyclic rank. For example, the graph at the root diameter might be circulant while the graph at the next diameter in its diameter class might be non-circulant Abelian Cayley with cyclic rank 2. The cyclic rank of a graph will be higher whenever the cyclic order and generator values for a cyclic dimension would otherwise have a greatest common divisor above 1.

6.8 Quasimaximal equivalence and properties

In this section, a final theorem is established to complete the frame of equivalence and properties of quasimaximal Abelian Cayley graph families and their lattice generator matrices (LGMs). In Section 6.4, Theorem 6.9 shows that if the canonical LGM of an

odd-degree Abelian Cayley graph family is radius maximal then the family is quasimaximal. We now establish the converse, that if a graph family is quasimaximal then its canonical LGM is radius maximal.

Theorem 6.17. *The canonical lattice generator matrix of a quasimaximal odd-degree Abelian Cayley graph family is radius maximal.*

Proof. We will prove that if the LGM is not radius maximal, then its Abelian Cayley graph family is not quasimaximal. Consider a lattice L in \mathbb{Z}^f generated by a radius-maximal canonical odd-degree LGM, $M = (\mathbf{v}_1 \dots \mathbf{v}_f)^T$, with involutory vector $\mathbf{v}_m = \sum \mathbf{v}_i/2$. Let L_p be the principal lattice generated by the vectors of M . Then its involutory lattice is $L_p + \mathbf{v}_m$, and $L = L_p \cup (L_p + \mathbf{v}_m)$. The corresponding lattice covering is comprised of Lee spheres of radius k centred on the principal lattice points and Lee spheres of radius $k - 1$ centred on the involutory lattice points. As M is radius maximal, $|\mathbf{v}_i - \mathbf{v}_m| = 2k$ for any i . Thus neighbouring pairs of Lee spheres of radius k and $k - 1$ abut precisely at an $(f - 1)$ -dimensional hyperplane parallel to their common faces. Each point in the neighbourhood is either within Manhattan distance k of the principal lattice point or within $k - 1$ of the involutory lattice point, but not both. There is no overlap nor gap between the faces of either sphere.

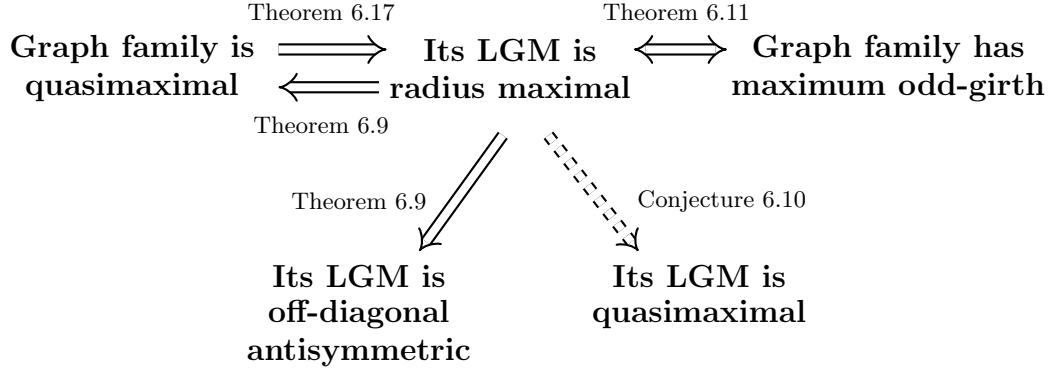
In contrast, if any radius is submaximal, then the corresponding faces of neighbouring Lee spheres will overlap at the $(f - 1)$ -dimensional cross-section to a depth equal to the extent of submaximality, assumed constant. Then the overlap for each involutory Lee sphere will have a volume of $ck^{f-1} + O(k^{f-2})$ for some constant c . This volume reduces the second coefficient of the order polynomial of the corresponding Abelian Cayley graph family, so that the family is no longer quasimaximal. \square

Combining Theorems 6.9, 6.11 and 6.17, we establish that the following three statements about an odd-degree Abelian Cayley graph family are equivalent.

- The graph family is quasimaximal
- The graph family has maximum odd girth
- Its LGM is radius maximal.

If any one of these three statements is true, then the other two are also true. In this case, its LGM is off-diagonal antisymmetric, and according to Conjecture 6.10 its LGM is quasimaximal. By translation between even and odd degrees of the same dimension, it is inferred that these relationships established for the LGM of odd-degree graph families apply equally to the LGM odd basis of even degree graph families. This is supported by the degree 6 investigation described in Section 6.5. These important relationships are presented graphically in Figure 6.10.

Figure 6.10: Summary of the main quasimaximal relationships for an Abelian Cayley graph family and its LGM (LGM odd basis, for even degree)



6.9 Relationships between families: translation, conjugation, transposition

There are three fundamental relationships between Abelian Cayley graph families of the same dimension: translation, conjugation and transposition. Translation between families was introduced and discussed in Section 5.5 and described in terms of canonical LGMs in Section 6.3. Conjugation between families was introduced in Section 5.4 and is explained in terms of LGMs and lattice coverings in this section. The third relationship, transposition, is then introduced. We begin with formal definitions of all three, followed by a summary with examples, see Table 6.3.

Definition 6.18. *Translation* is a relationship between pairs of Abelian Cayley graph families of the same odd dimension f , one of even degree and one of odd, defined in terms of their canonical LGMs. They belong to different diameter classes. If the even-degree family has diameter class root $k_{\mathcal{X}}$ and the odd-degree $k_{\mathcal{Y}}$, then we have $k_{\mathcal{X}} = (k_{\mathcal{Y}} + (f - 1)/2) \bmod f$. Such a pair is related by *translation* if the LGM of the odd-degree family is the LGM odd basis of the even-degree family. They are said to *translate* of each other.

Definition 6.19. *Conjugation* is a relationship between pairs of quasimaximal Abelian Cayley graph families of any dimension f and the same degree, defined in terms of their canonical LGMs. They belong to different diameter classes unless the class is principal. If their diameter class roots are $k_{\mathcal{X}}$ and $k_{\mathcal{Y}}$, then

$$k_{\mathcal{X}} + k_{\mathcal{Y}} = \begin{cases} f - 1 & \text{for even degree} \\ f & \text{for odd degree.} \end{cases}$$

Such a pair is related by *conjugation* if the LGM of one family (LGM odd basis, for even degree) is equal to the matrix obtained by reversing the sign of the constant in each diagonal element of the other LGM. The families are said to be *conjugates* of each other. Conjugate pairs in the principal diameter class are called *self-conjugate* if they are isomorphic.

Definition 6.20. *Transposition* is a relationship between pairs of Abelian Cayley graph families of the same degree and diameter class, defined in terms of their canonical LGMs. Such a pair is related by *transposition* if the canonical LGM of one family (LGM odd basis, for even degree) is equal to the transpose matrix of the other LGM. The families are said to be *transposes* of each other. If the pair are isomorphic, they are called *self-transpose*.

Table 6.3: Summary of translation, conjugation and transposition, with an example of each

Translation	Conjugation	Transposition
An odd-degree LGM is the even-degree LGM odd basis of its translate graph family	The conjugate of an odd-degree LGM (or even-degree LGM odd basis) is obtained by reversing the sign of the constant in each diagonal element	The transpose matrix of an odd-degree LGM (or even degree LGM odd basis) is the LGM of the transpose graph family
Between different degrees and different diameter classes	Between different diameter classes of the same degree (unless principal)	Between families of the same degree and diameter class
Only for odd dimensions	Only for quasimaximal families	
	Families of the principal diameter class may be self-conjugate	Families may be self-transpose
F7:1A LGM $(d = 7, k \equiv 1 \pmod{3})$ $\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	F7:1A LGM $(d = 7, k \equiv 1 \pmod{3})$ $\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	F9:1a LGM $(d = 9, k \equiv 1 \pmod{2})$ $\begin{pmatrix} 2a-2 & 1 & 0 & 1 \\ -1 & 2a-1 & 1 & 1 \\ 0 & -1 & 2a-1 & 0 \\ -1 & -1 & 0 & 2a \end{pmatrix}$
F6:2A LGM odd basis $(d = 6, k \equiv 2 \pmod{3})$ $\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	F7:2A LGM $(d = 7, k \equiv 2 \pmod{3})$ $\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$	F9:1b LGM $(d = 9, k \equiv 1 \pmod{2})$ $\begin{pmatrix} 2a-2 & -1 & 0 & -1 \\ 1 & 2a-1 & -1 & -1 \\ 0 & 1 & 2a-1 & 0 \\ 1 & 1 & 0 & 2a \end{pmatrix}$

We have proved in Theorem 5.8 the validity of translation as defined. Unfortunately it has not yet been possible to prove the validity of conjugation and transposition separately, and so these are documented as conjectures. We now prove the validity of the combination of conjugation and transposition, in other words, that every quasimaximal Abelian Cayley graph family has a *conjugate transpose*.

Theorem 6.21. *Let $\mathcal{X} = \{X(d, k) : k \in K\}$ be a quasimaximal odd-degree Abelian Cayley graph family of dimension f for some diameter class K with root diameter $k_{\mathcal{X}}$. Let its canonical LGM be M with leading diagonal $(2a + b_1, \dots, 2a + b_f)$ where $a = (2k + c)/f$ with c chosen such that a is an integer for all $k \in K$. Let M' be constructed from M by replacing the leading diagonal with $(2a - b_1, \dots, 2a - b_f)$ and then taking the transpose matrix. Then M' is the canonical LGM for a quasimaximal odd-degree Abelian Cayley graph family of dimension f for diameter class K' with root diameter $k'_{\mathcal{X}} = f - k_{\mathcal{X}} \pmod f$. This family is called the conjugate transpose of \mathcal{X} .*

Proof. By Theorem 6.17, we may assume that M is radius maximal. Then for any value of the parameter a corresponding to a diameter $k \in K$, each pair of parallel $(f - 1)$ -dimensional hyperfaces of adjacent Lee spheres in the associated lattice covering of \mathbb{Z}^f abut precisely without overlap or gap. This property remains true when a takes a negative value as the distance between the pairs of hyperfaces is a constant and so does not depend on a . Hence M remains radius maximal for negative a . We now define $M' = -M$ to obtain a matrix in canonical LGM format. As M' generates a lattice covering and is radius maximal, by Theorem 6.9 it is the LGM for a quasimaximal graph family. \square

The conjecture that every quasimaximal Abelian Cayley graph family has a conjugate family, as defined by Definition 6.20, has already been stated in Chapter 5 as Conjecture 5.7.

Conjecture 6.22. *Every Abelian Cayley graph family has a transpose family as defined by Definition 6.20.*

It is immediate from Theorem 6.21 that Conjectures 5.7 and 6.22 stand or fall together. Proving the validity of either would prove the other.

CHAPTER 7

ENUMERATION OF A CLASS OF DEGREE 7 CIRCULANT GRAPH FAMILIES

In this chapter, we explore the correlation between families of maximum-odd-girth circulant graphs and quasimaximal families and also their correlation with canonical LGMs. The analysis was conducted for degree 7 and diameter class 0 (mod 3), being a large enough degree to have interesting structure whilst being small enough to be tractible by computer. What emerges in this case is a compelling example of the coincidence of those families that are quasimaximal and those with maximum odd girth, and also of a bijection between those categories of graph families and the set of matrices with canonical LGM format.

A computer search was conducted to discover the range of order of degree 7 circulant graphs of diameter class 0 for diameters 3, 6, 9, 12, 15 and 18. Considering all diameter 18 graphs found, the largest has order 13,896 and has maximum odd girth, 37. The largest graph with lower odd girth has order 13,360 and odd girth 35. The smallest maximum-odd-girth graph covered by the search has order 11,832, an arbitrary limit. This pattern is repeated across all six diameters, with the order of the largest lower-odd-girth graph exceeding the largest submaximal one, which exceeded other maximum-odd-girth ones.

There happen to be 58 graphs of diameter 18 with maximum odd girth with distinct order between 11,832 and 13,896. They all belong to quasimaximal families that include the largest 58 graphs of diameter 15 with maximum odd girth, and so on. By contrast, the lower-odd-girth graphs of diameter 18 with order between 11,832 and 13,360 all belong to subquasimaximal families. This analysis supports the fact that the graphs in quasimaximal families all have maximum odd girth, Theorem 6.16.

Degree 7 families with largest order are listed in Table 7.1 along with the largest order family with lower odd girth for comparison.

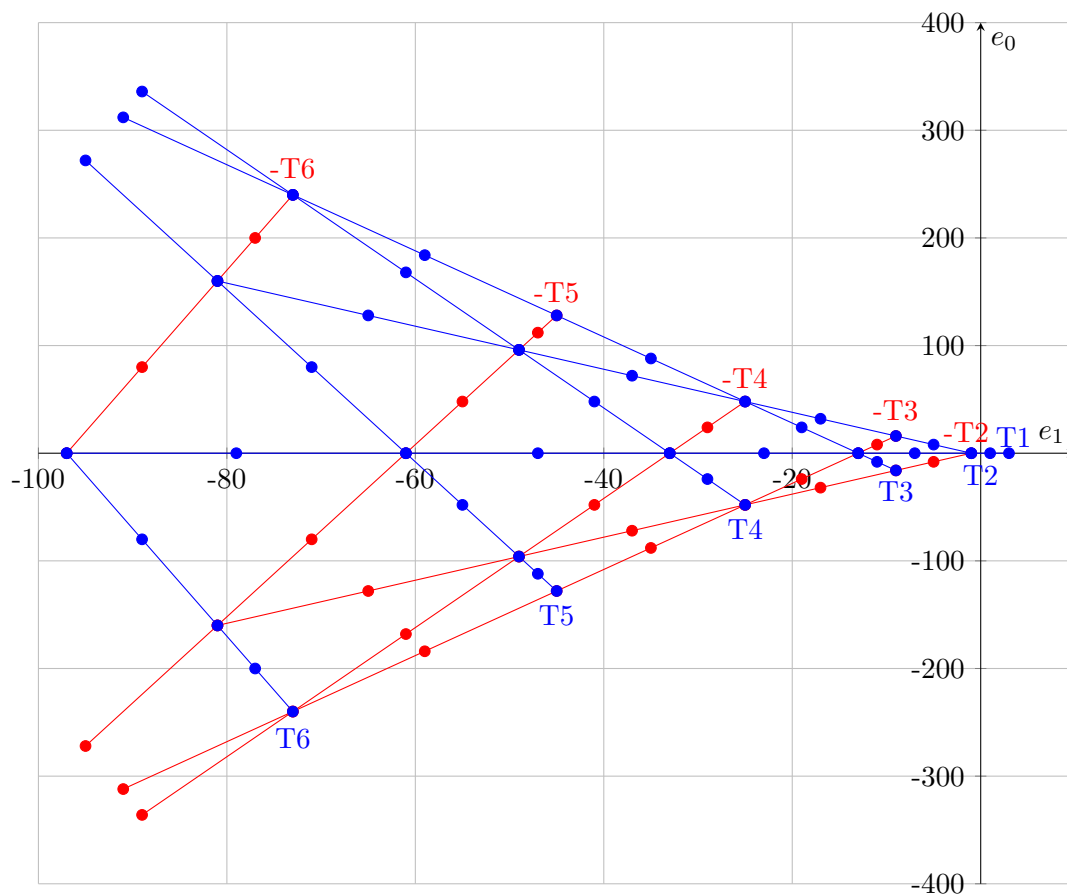
Considering a general cubic in vector format as $(e_3 \ e_2 \ e_1 \ e_0)$, for the order of any degree 7 quasimaximal family of diameter class 0 (mod 3), we have $e_3 = 1$ and $e_2 = 0$. So each of the 58 largest degree 7 quasimaximal circulant graph families are uniquely

Table 7.1: Order of eight largest degree 7 circulant graph families

Odd girth defect	Diameter, k						Order polynomial*
	3	6	9	12	15	18	
0	76	536	1764	4144	8060	13896	$(1 \ 0 \ 3 \ 0)$
0	68	520	1740	4112	8020	13848	$(1 \ 0 \ 1 \ 0)$
0	60	504	1716	4080	7980	13800	$(1 \ 0 \ -1 \ 0)$
0	52	480	1676	4024	7908	13712	$(1 \ 0 \ -5 \ 8)$
0	36	464	1660	4008	7892	13696	$(1 \ 0 \ -5 \ -8)$
0	36	456	1644	3984	7860	13656	$(1 \ 0 \ -7 \ 0)$
0	44	456	1636	3968	7836	13624	$(1 \ 0 \ -9 \ 16)$
0	12	424	1604	3936	7804	13592	$(1 \ 0 \ -9 \ -16)$
2	60	480	1636	3912	7692	13360	$(1 \ -1 \ 5 \ -8)$

* Cubic in $2a$, where $a = 2k/3$

defined by their values for e_1 and e_0 and can be positioned on a chart with these two axes; see Figure 7.1.

Figure 7.1: Parameters e_1 and e_0 for largest degree 7 quasimaximal circulant graph families

Each blue or red point on the chart represents a degree 7 quasimaximal circulant graph family, and all other such families have $e_1 < -100$ and so lie to the left of the range of the chart. Some pattern is evident in this chart, with the points aligned in blue and red diagonal threads as illustrated. Each thread runs with decreasing e_1

value from its first member. The first thread, T1, has first member at position (3,0) representing the largest-known degree 7 circulant graph family of diameter class 0 with order polynomial (1 0 3 0). The thread runs flat along the horizontal axis, so that $e_0 = 0$ for every member. Threads T2 to T6 are shown in blue. They are mirrored by red threads, denoted -T2 to -T6, where the sign of e_0 is reversed for each member. Some families are members of a single thread, others of two or three threads. Obvious questions to ask are why the quasimaximal degree 7 circulant graph families should have parameters at these points and no others, and why the points should be aligned in threads in this way. To answer these questions, we approach the matter from an alternative perspective by considering all possible LGMs for quasimaximal degree 7 circulant graph families of diameter class 0 (mod 3).

The canonical quasimaximal degree 7 LGM has the following format, by definition:

$$M = \begin{pmatrix} 2a + b_1 & c_{12} & c_{13} \\ -c_{12} & 2a + b_2 & c_{23} \\ -c_{13} & -c_{23} & 2a + b_3 \end{pmatrix}$$

subject to constraints. The trace equals $6a$, so that $b_1 + b_2 + b_3 = 0$. Each c_{ij} has value 0 or ± 1 . The column totals are all even, so that, for instance, $b_1 - c_{12} - c_{13}$ is even. We will also impose an additional constraint that each column must contain at least one non-zero element. The reason for this is discussed later, but its effect is to exclude Abelian Cayley graph families that are not circulant. Without loss of generality, we also choose to order the columns so that the leading diagonal is either $(2a + b_1, 2a - b_2, 2a - b_3)$, denoted $[b_1, b_2]$, or $(2a - b_1, 2a + b_2, 2a + b_3)$, denoted $-[b_1, b_2]$, where $0 \leq b_2 \leq b_3 \leq b_1$. Then the zero elements are uniquely determined by the parity of the diagonal elements, giving four distinct formats:

$$\begin{pmatrix} \text{even} & c_{12} & c_{13} \\ -c_{12} & \text{even} & c_{23} \\ -c_{13} & -c_{23} & \text{even} \end{pmatrix}, \begin{pmatrix} \text{even} & c_{12} & c_{13} \\ -c_{12} & \text{odd} & 0 \\ -c_{13} & 0 & \text{odd} \end{pmatrix}, \begin{pmatrix} \text{odd} & c_{12} & 0 \\ -c_{12} & \text{even} & c_{23} \\ 0 & -c_{23} & \text{odd} \end{pmatrix}, \begin{pmatrix} \text{odd} & 0 & c_{13} \\ 0 & \text{odd} & c_{23} \\ -c_{13} & -c_{23} & \text{even} \end{pmatrix}$$

where $|c_{12}| = |c_{13}| = |c_{23}| = 1$.

It is clear that the last three are self transpose. However, the first represents two distinct transpose cases. Therefore, from the specification of the identifier, $\pm[b_1, b_2]$, and dependent on the parity of b_1 and b_2 , the order of the graphs in the family with this LGM is uniquely defined as a cubic in $2a$, assuming such a family exists.

Representing the cubic in vector format as $(e_3 \ e_2 \ e_1 \ e_0)$, we immediately have $e_3 = 1$ and $e_2 = 0$, as the LGM format is quasimaximal. For e_1 , we have the general equation: $e_1 = \sum_{i < j} b_i b_j - \sum_{i \neq j} c_{ij} c_{ji}$.

Note that if the sign of all the b_i are reversed then the result is unchanged. Also, reversing the sign of any c_{ij} leaves the results unchanged as the product of the pairs is always -1 in this case. Hence, for any b_1 and b_2 , the value of e_1 for $[b_1, b_2]$ is the same as for $-[b_1, b_2]$. In contrast, although the magnitude of e_0 remains unchanged, its sign is reversed. See Table 7.2.

Table 7.2: Formulae for e_1 and e_2 in $\pm[b_1, b_2]$

b_1	b_2	b_3	e_1	e_0 (negative for $-[b_1, b_2]$)
even	even	even	$3 - b_2^2 - b_1 b_3$	$b_1 b_2 (b_1 - b_2)$
even	odd	odd	$2 - b_2^2 - b_1 b_3$	$b_1 b_2 (b_1 - b_2) - b_1$
odd	even	odd	$2 - b_2^2 - b_1 b_3$	$b_1 b_2 (b_1 - b_2) + b_2$
odd	odd	even	$2 - b_2^2 - b_1 b_3$	$b_1 b_2 (b_1 - b_2) + b_1 - b_2$

By definition, $0 \leq b_2 \leq \lfloor b_1/2 \rfloor$. So we have a simple enumeration of the canonical LGMs as an infinite sequence: $[0, 0], [1, 0], [2, 0], [2, 1], -[2, 1], [3, 0], [3, 1], -[3, 1], \dots$. Of course, there is no a priori reason why any of these matrices should actually be the LGM of an existent graph family. By illustration, some examples are presented in Table 7.3.

Table 7.3: Some canonical degree 7 quasimaximal LGMs and the order of the graphs in their families, if they exist

$[b_1, b_2]$	LGM	Order, in $2a$ where $a = 2k/3$
$[0, 0]$	$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix},$	$(1 \ 0 \ 3 \ 0)$
$[1, 0]$	$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix},$	$(1 \ 0 \ 1 \ 0)$
$[3, 1]$	$\begin{pmatrix} 2a+3 & 0 & -1 \\ 0 & 2a-1 & -1 \\ 1 & 1 & 2a-2 \end{pmatrix},$	$(1 \ 0 \ -5 \ 8)$
$-[3, 1]$	$\begin{pmatrix} 2a-3 & 0 & -1 \\ 0 & 2a+1 & -1 \\ 1 & 1 & 2a+2 \end{pmatrix},$	$(1 \ 0 \ -5 \ -8)$

If we take the LGMs denoted by $\pm[b_1, b_2]$ where $0 \leq b_1 \leq 10$ and plot them on a chart by the values of e_1 and e_0 in their corresponding order cubics, then the points coincide exactly with the points representing the largest families as shown in Figure 7.1. Over the relatively wide range of quasimaximal degree 7 families and LGMs investigated, there is a one-to-one correlation between the two categories. This provides answers to the earlier questions about the reason that only degree 7 quasimaximal families with

particular parameter sets exist: they are precisely the families that correspond to LGMs in canonical quasimaximal format. This relation supports an enumeration of all degree 7 quasimaximal circulant graph families of diameter class 0.

The thread T1 runs along the horizontal axis of the chart, containing $[0, 0], [0, 1], [2, 0], \dots$; T2 contains $[2, 0], [3, 1], [4, 2], \dots$ and its reflection -T2 contains $[1, 0], [-[3, 1], -[4, 2], \dots$; etc. See Table 7.4

Table 7.4: The first six members of the first three threads

Position in thread	Thread T1			Thread T2			Thread T3		
	LGM notation	Order	coeffs e_1 e_0	LGM notation	Order	coeffs e_1 e_0	LGM notation	Order	coeffs e_1 e_0
1	$[0, 0]$	3	0	$[2, 0]$	-1	0	$-[4, 2]$	-9	-16
2	$[1, 0]$	1	0	$[3, 1]$	-5	8	$-[4, 1]$	-11	-8
3	$[2, 0]$	-1	0	$[4, 2]$	-9	16	$[4, 0]$	-13	0
4	$[3, 0]$	-7	0	$[5, 2]$	-17	32	$[5, 1]$	-19	24
5	$[4, 0]$	-13	0	$[6, 2]$	-25	48	$[6, 2]$	-25	48
6	$[5, 0]$	-23	0	$[7, 2]$	-37	72	$[7, 3]$	-35	88

For graph families in thread T1, with order polynomial $(1 \ 0 \ e_1 \ e_0)$, the value of e_1 and e_0 may be calculated from the position p of the graph family in the thread as follows:

$$e_1 = \begin{cases} -p^2 + 2p + 2, & \text{for odd } p \\ -p^2 + 2p + 1, & \text{for even } p \end{cases} \quad e_0 = 0.$$

Similarly for thread T2:

$$e_1 = \begin{cases} -p^2, & \text{for odd } p \\ -p^2 - 1, & \text{for even } p \end{cases} \quad e_0 = \begin{cases} 2p^2 - 2, & \text{for odd } p \\ 2p^2, & \text{for even } p. \end{cases}$$

The formulae in Table 7.5 enable the order of the graph family in any position in any thread to be determined. A similar analysis would also provide formulae for their generating sets.

Table 7.5: Formulae for the order of quasimaximal circulant graph families of degree 7 and diameter class 0

Thread	Position	Linear coefficient		Constant term
t	p	e_1		e_0
Odd	Odd	$-p^2 + 2p - 3t^2 + 6t - 1$		$(2t - 2)p^2 - (4t - t)p - 2t^3 + 6t^2 - 4t$
Odd	Even	$-p^2 + 2p - 3t^2 + 6t - 2$		$(2t - 2)p^2 - (4t - 4)p - 2t^3 + 6t^2 - 2t - 2$
Even	Odd	$-p^2$	$-3t^2 + 6t$	$(2t - 2)p^2 - 2t^3 + 6t^2 - 6t + 2$
Even	Even	$-p^2$	$-3t^2 + 6t - 1$	$(2t - 2)p^2 - 2t^3 + 6t^2 - 4t$

As mentioned, this equivalence between matrices in canonical degree 7 quasimaximal LGM format for diameter class 0, with the additional constraints, on the one hand,

and degree 7 quasimaximal circulant graph families of diameter class 0, on the other, is a bijection within the range investigated. There are no gaps or exceptions.

You may remember that we imposed an additional constraint that each column in an LGM had to include at least one non-zero off-diagonal element. If this constraint is relaxed, then it is possible to have LGMs with one or three columns with all-zero off-diagonal elements. In the same way that LGMs with no such columns correspond to circulant graph families, it emerges that LGMs with one such column correspond to Abelian Cayley graph families of cyclic rank 2, and those with three, to Abelian Cayley graph families of cyclic rank 3. These also appear to be bijections. The foregoing relates to quasimaximal graph families of degree 7 and diameter class 0. It is conjectured that the same bijection exists for the other two diameter classes, and equally for degree 6.

Conjecture 7.1. *For degree 7 and each of the three diameter classes (0, 1 and 2 mod 3) there is a bijection between quasimaximal circulant graph families and matrices in canonical quasimaximal LGM format with at least one non-zero off-diagonal element in each column. There are also bijections between quasimaximal Abelian Cayley graphs of cyclic rank 2 and 3 and matrices in canonical quasimaximal LGM format with one and three columns with all-zero off-diagonal elements respectively. Similarly, for degree 6 and each of the three diameter classes, there are bijections between the same categories of graph families and matrices in the format of canonical quasimaximal LGM odd basis respectively.*

On the other hand, we have already seen in Section 6.4 an example for degree 9 of a matrix in canonical quasimaximal LGM format that is not the LGM of a graph family. It is conjectured that for dimension 4 and above, the canonical quasimaximal LGM format admits matrices where the distance between a pair of neighbouring vertices of the lattice unit cell exceeds $2k + 1$ where k is the diameter, and that these matrices are not LGMs for Abelian Cayley graph families.

Conjecture 7.2. *For each degree greater than or equal to 8, there are matrices in canonical quasimaximal LGM format that are not the LGM for an Abelian Cayley graph family.*

LARGEST-KNOWN CIRCULANT GRAPH FAMILIES OF DEGREES 12 TO 20

Chapter 8 presents newly discovered largest-known circulant graph families up to degree 20, all quasimaximal. The efficient search for these families depended on LGM properties discussed in Chapter 6. Many of these families are related to others by the relationships described in Chapter 5. The interaction of the three relationships - translation, conjugation and transposition - for the largest-known circulant graph families of each dimension is presented graphically within a *dimensional frame*. As defined by Definition 1.4, a graph family is an infinite set of graphs of given degree d and dimension $f = \lfloor d/2 \rfloor$, defined for each diameter k of a diameter class, with order and generating set specified by polynomials in k of maximum degree f .

8.1 Improved search strategy for graph families

The largest-known circulant graph families up to degree 11 were found by increasingly time-consuming computer search, with individual trials running for up to a month in some cases. Despite various optimisation assumptions being made and algorithmic efficiencies adopted, it was clear that this approach had reached practical limits. In order to discover a graph family by this approach, it is necessary first to discover candidate graphs for a set of diameters equal to the dimension, as this is the order of the target polynomial. For each diameter, this requires running the computer search program for each order within a reasonable range of values.

A rough order of magnitude for the number of potential generating sets to be checked for each order n of a graph of dimension f can be obtained from the number of combinations of size f that can be chosen from $n/2$ elements (as generators can be taken as the lower of each pair of connection elements). Each combination will duplicate its automorphic equivalents, and this is compensated by dividing by $n/2$. A simple calculation reveals the challenge for a degree 12 example, assuming the need to discover candidate graphs for diameters 3, 6, 9, 12, 15 and 18. Table 8.1 shows the net number of combinations of generating sets to be tested for graphs with the orders of the largest-known family.

Table 8.1: Example; degree 12, diameter class 0 (mod 3). Number of trial graphs to be evaluated for each order at each diameter

Diameter	Order	Number of trials
3	240	4×10^9
6	5,044	4×10^{17}
9	39,996	9×10^{22}
12	190,392	1×10^{27}
15	662,680	2×10^{30}
18	1,868,940	9×10^{32}

This rapidly becomes an impossibly large number of combinations to handle.

Moreover, these need to be multiplied by the number of trial orders to be investigated.

A more extreme example is provided by such a search for a graph family of degree 18, diameter class 0 (mod 9), requiring searches for diameters 9, 18, 27, 36, 45, 54, 63, 72 and 81. The largest-known circulant graph of degree 18 and diameter 81 has order 54,541,109,677,608, over 54 trillion. For this diameter, the number of potential generating sets to be tested for each order within the specified search range is about 10^{101} , an impossible task.

Searching for large graph families using candidate lattice generator matrices reduces the computational effort dramatically for two reasons. Firstly, the degrees of freedom for each degree and diameter class is determined by the dimension and does not increase with diameter. Secondly, the search is for complete families and not for individual graphs which need to be matched into families manually.

By restricting the search to canonical lattice generator matrices for quasimaximal graph families (their odd bases for even degree), we benefit from the restrictions that the trace is fixed, the eccentricity is 0, and the column totals are all even. For any dimension f and any chosen leading diagonal (in practice, limited to few alternatives), the number of candidate lattice generating vectors to be evaluated is roughly $3^{f(f-1)/2}/2^{f-1}$. For degree 12, $f = 6$ and the number is about 450,000. For degree 18, it is 5×10^{14} . Although large numbers, these are small in comparison with the number of trial graphs in the direct search method. Also, the lattice generator matrix search method does not require trial graph orders to be conjectured in advance. And of course, the results are whole families, complete with orders and generating sets. This is the method that was used in the search for largest-known circulant graph families of degrees 12 to 20. It was also used subsequently in a less-exhaustive search for general Abelian Cayley graph families up to degree 15, which we will review in Chapter 10.

The mechanism to produce graph families from LGMs is essentially a reversal of the steps taken by Dougherty and Faber. We need to solve a set of simultaneous

equations defined by the lattice vectors, the rows of the matrix. In principle this would be achieved by first inverting the matrix. However, for larger dimension, inversion can cause rounding errors, especially for elements with magnitude close to zero. In order to avoid these errors, the calculations are maintained within the ring of integers by taking the adjoint of the LGM instead of the inverse (where *adjoint* has the classical definition as the transpose of the cofactor matrix). After other calculations have been completed, the values obtained are divided by the determinant to achieve the desired results.

Candidate generating sets are produced by post-multiplying the adjoint by a column vector. The algorithm initially loops through combinations of elements of magnitude 0 and 1 in the column vector to find candidate sets that have no zero values and no duplicates. Selected sets are then tested directly to see if the resultant graph has the required diameter. If necessary, the column vector elements may progressively include 2, 3, etc., until solutions are found.

For each largest-known circulant graph family of dimension f , we present its lattice generator matrix (LGM odd basis, for even degree), and its order and a generating set, expressed as polynomials in vector notation in the variable $2a$, where a is defined in terms of the diameter k as $a = (2k + c)/f$ for a suitable constant c . In order not to interrupt the flow, only one diameter class, $k \equiv 0 \pmod{f}$, is included here for each degree, as an example. The full set of families is presented in Appendix A, and details of each graph can be found in Appendix D. The existence of these families for all diameters in their respective diameter classes is confirmed by the Existence Proof Theorem for Abelian Cayley graph families, Theorem 6.16.

8.2 Circulant graph dimensional frames

By considering transposition, conjugation and translation, it is possible for up to eight distinct isomorphism classes of Abelian Cayley graph families to be associated with a single canonical lattice generator matrix. We will consider a canonical odd-degree LGM of arbitrary dimension and an associated Abelian Cayley graph family. If the dimension is odd, there will be an even-degree translate of the family. If the LGM is quasimaximal, both families may have conjugate families of the same degree but different diameter class (unless the LGM is self-conjugate). The (possibly) four families may each have a transpose family (unless the LGM is self-transpose). Any of these three factors may apply, giving a total of 1, 2, 4 or 8 distinct isomorphism classes.

As an example, for dimension 9 there are eight such families, all largest-known, associated with a unique LGM. We define four 9×9 matrices (two diagonal and an antisymmetric pair):

$$D_- = \text{Diag}(2a, 2a, 2a - 1, 2a - 1, 2a - 1, 2a - 1, 2a - 1, 2a - 1, 2a - 1)$$

$$D_+ = \text{Diag}(2a, 2a, 2a + 1, 2a + 1, 2a + 1, 2a + 1, 2a + 1, 2a + 1, 2a + 1)$$

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } A^T.$$

Note that the sum of a D and an A has the format of a canonical quasimaximal degree 19 LGM. By conjugation and transposition we have four degree 19 largest-known circulant graph families, and by translation we have four degree 18 largest-known families. These are listed in Table 8.2.

Table 8.2: Eight circulant graph families associated with a single LGM

Conjugation and transposition					
LGM	Degree	Diameter class	Order polynomial (in $2a$)	where $a =$	Family
$D_- + A$	19	7 (mod 9)	(1 -8 50 -194 462 -698 672 -394 125 -16)	$(2k + 4)/9$	F19:7a
$D_- + A^T$	19	7 (mod 9)	(1 -8 50 -194 462 -698 672 -394 125 -16)	$(2k + 4)/9$	F19:7b
$D_+ + A$	19	2 (mod 9)	(1 8 50 194 462 698 672 394 125 16)	$(2k - 4)/9$	F19:2a
$D_+ + A^T$	19	2 (mod 9)	(1 8 50 194 462 698 672 394 125 16)	$(2k - 4)/9$	F19:2b
Also translation					
LGM odd basis	Degree	Diameter class	Order polynomial (in $2a$)	where $a =$	Family
$D_- + A$	18	2 (mod 9)	(1 -8 50 -194 462 -698 672 -394 125 -16)/2	$(2k + 5)/9$	F18:2a
$D_- + A^T$	18	2 (mod 9)	(1 -8 50 -194 462 -698 672 -394 125 -16)/2	$(2k + 5)/9$	F18:2b
$D_+ + A$	18	6 (mod 9)	(1 8 50 194 462 698 672 394 125 16)/2	$(2k - 3)/9$	F18:6a
$D_+ + A^T$	18	6 (mod 9)	(1 8 50 194 462 698 672 394 125 16)/2	$(2k - 3)/9$	F18:6b

For each dimension, these relationships between largest-known circulant graph families, where they exist, may be shown graphically by positioning the families suitably within a *dimensional frame*, see Table 8.3.

Table 8.6: Dimension 5: largest-known circulant graph families

Degree 10 families			Degree 11 families		
F10:0	←			→	F11:3
↑	F10:1			F11:4	↑
↓	↕	F10:2	↔	F11:0	↓
	F10:3			F11:1a/b	
F10:4	←			→	F11:2

8.3 Dimension 6, degrees 12 and 13

As the dimension is even, there is no translation of graph families between degrees 12 and 13; they are essentially two independent graph family solution spaces. Also, as the dimension is even, the graph families are defined for diameter classes modulo $h = f/2 = 3$. Thus, there are three classes of largest-known circulant graph families for each degree.

For degree 12, for diameter class 0 there is one self-transpose family. For diameter classes 1 and 2, there are transpose pairs of non-isomorphic families. The families for classes 0, and 2 are not conjugates because the conjugate of each class has lower order polynomial. For instance, the order of F12:0 is $(1\ 2\ 11\ 14\ 13\ 6\ 0)/2$ and so has conjugate in class 2 with order $(1\ -2\ 11\ -14\ 13\ -6\ 0)/2$. However, the order of F12:2 is $(1\ -2\ 11\ -12\ -2\ 4\ 0)/2$, which has a larger (less negative) fourth coefficient.

For degree 13, for all three diameter classes there are transpose pairs of non-isomorphic families. Moreover, the families for classes 1 and 2 are conjugates of each other, see Table 8.7.

Table 8.7: Order of largest-known circulant graph families of degrees 12 and 13

Degree 12 families		Degree 13 families		
F12:	Order polynomial in $2a$	F13:	Order polynomial in $2a$	c^*
0	$(1\ 2\ 11\ 14\ 13\ 6\ 0)/2$	0a/b	$(1\ 0\ 8\ 2\ -1\ -4\ 0)$	0
1a/b	$(1\ 6\ 24\ 58\ 75\ 46\ 10)/2$	1a/b	$(1\ 4\ 16\ 30\ 29\ 16\ 4)$	-1
2a/b	$(1\ -2\ 11\ -12\ -2\ 4\ 0)/2$	2a/b	$(1\ -4\ 16\ -30\ 29\ -16\ 4)$	1
* for degree 12, $a = (k + c)/3$		* for degree 13, $a = (k + c)/3$		

These relationships are summarised in the dimensional frame, Table 8.8.

Table 8.8: Dimension 6: largest-known circulant graph families

Degree 12 families		Degree 13 families	
F12:0			F13:2a/b
	F12:1a/b	F13:0	↕
F12:2a/b			F13:1a/b

For diameter class 0, the LGM odd bases and formulae for the order and a generating set for degree 12 are shown in Table 8.9, and the LGMs and formulae for degree 13 in Table 8.10. A full set, for all diameter classes, is given in Appendix A.5, and properties of the individual graphs are shown in Appendix D.

Table 8.9: Degree 12, diameter class 0, $a = k/3$

LGM odd basis						Polynomial in $2a$							
Family F12:0 (self-transpose)						Order	(1	2	11	14	13	6	0) /2
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 2a+1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 2a & 1 & -1 & 0 \\ 1 & 0 & -1 & 2a & -1 & -1 \\ 0 & 1 & 1 & 1 & 2a & 1 \\ 0 & 0 & 0 & 1 & -1 & 2a \end{pmatrix}$						g_1	(0	2	4	24	16	13	0) /2
						g_2	(0	0	1	-3	-5	-9	0) /2
						g_3	(0	2	3	14	8	9	2) /2
						g_4	(0	0	2	1	0	1	-2) /2
						g_5	(0	1	-1	2	-6	-5	-2) /2
						g_6	(0	1	3	8	15	7	0) /2

Table 8.10: Degree 13, diameter class 0, $a = k/3$

LGM						Polynomial in $2a$							
Family F13:0a (transpose of F13:0b)						Order	(1	0	8	2	-1	-4	0)
$\begin{pmatrix} 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 2a+1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 & -1 \\ 0 & 0 & 1 & 2a & 1 & 0 \\ 1 & 1 & 1 & -1 & 2a-1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$						g_1	(0	1	1	0	1	-2	-1)
						g_2	(0	1	-2	3	-1	-2	-1)
						g_3	(0	0	2	8	3	0	-1)
						g_4	(0	1	1	3	-2	0	-1)
						g_5	(0	1	-3	4	4	3	-1)
						g_6	(0	1	3	4	5	4	1)
Family F13:0b (transpose of F13:0a)						Order	(1	0	8	2	-1	-4	0)
$\begin{pmatrix} 2a+1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2a+1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 2a & 1 & 1 & 1 \\ 0 & 0 & -1 & 2a & -1 & 0 \\ -1 & -1 & -1 & 1 & 2a-1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 2a-1 \end{pmatrix}$						g_1	(0	1	-2	2	-2	0	1)
						g_2	(0	1	2	1	5	-4	-1)
						g_3	(0	1	3	3	0	0	-1)
						g_4	(0	0	0	6	1	-4	-1)
						g_5	(0	1	-3	2	4	1	-1)
						g_6	(0	1	0	0	0	-2	-1)

8.4 Dimension 7, degrees 14 and 15

As the dimension is odd, there is translation between degree 14 graph families of odd order and degree 15 families. Also, as the dimension is odd, the graph families are defined for diameter classes modulo $f = 7$, so there are seven classes of largest-known circulant graph families for each degree.

For degree 14, for four of the seven diameter classes there are transpose pairs of non-isomorphic families. They also form two sets of conjugate pairs. However, as they all have even order, they translate to degree 15 Abelian Cayley graph families of cyclic rank 2 rather than circulant graph families. For the other three diameter

classes, the families all have odd order and thus have translates in degree 15. They are self-transpose. Two form a conjugate pair. The other is self-conjugate and also has a multiplicative generating set, so that it is arc-transitive.

For degree 15, there is a transpose pair of non-isomorphic families only for diameter class 3. Their conjugates are not largest-known families and nor are their translates. For the other six diameter classes, the families are all self-transpose. Two form a conjugate pair with translates that are largest-known degree 14 families. Another two form a conjugate pair with translates that are not largest-known. For the fifth, neither its conjugate nor its translate is largest-known. The final self-transpose family is self-conjugate and its translate is the arc-transitive degree 14 family, see Table 8.11.

Table 8.11: Order of largest-known circulant graph families of degrees 14 and 15

Degree 14 families		Degree 15 families		c*
F14:	Order polynomial in $2a$	F15:	Order polynomial in $2a$	
0a/b	(1 2 15 20 21 12 4 0)/2	4	(1 2 14 20 27 18 11 2)	0
1	(1 6 28 76 127 126 67 14)/2	5	(1 6 28 76 127 126 67 14)	-2
2a/b	(1 -4 21 -46 50 -30 8 0)/2	6	(1 -4 20 -44 57 -44 19 -4)	3
3	(1 0 14 0 21 0 7 0)/2	0	(1 0 14 0 21 0 7 0)	1
4a/b	(1 4 21 46 50 30 8 0)/2	1	(1 4 20 44 57 44 19 4)	-1
5	(1 -6 28 -76 127 -126 67 -14)/2	2	(1 -6 28 -76 127 -126 67 -14)	4
6a/b	(1 -2 15 -20 21 -12 4 0)/2	3a/b	(1 -2 14 -16 11 -6 3 -2)	2

* for degree 14, $a = (2k + c)/7$ * for degree 15, $a = (2k + c - 1)/7$

These relationships are summarised in the dimensional frame, Table 8.12.

Table 8.12: Dimension 7: largest-known circulant graph families

Degree 14 families				Degree 15 families	
F14:0a/b					F15:4
	F14:1	←			F15:5
		F14:2a/b		F15:6	
		↕	F14:3 ↔ F15:0	↕	F15:1
	F14:4a/b				F15:2
	F14:5	←			
F14:6a/b					F15:3a/b

For diameter class 0, the LGMs and formulae for degrees 14 and 15 are shown in Tables 8.13 and 8.14, with a full set in Appendix A.6. Properties of the individual graphs are given in Appendix D.

Table 8.13: Degree 14, diameter class 0, $a = 2k/7$

LGM odd basis							Polynomial in $2a$							
Family F14:0a (transpose of F14:0b, conjugate of F14:6a)														
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 2a & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2a & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 2a & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order	(1	2	15	20	21	12	4	0)	/2				
	g_1	(0	0	1	3	-2	-1	-4	0)	/2				
	g_2	(0	0	1	-2	3	-2	-2	0)	/2				
	g_3	(0	0	1	6	14	15	8	0)	/2				
	g_4	(0	0	0	4	6	-1	0	0)	/2				
	g_5	(0	0	1	-1	-3	-5	-3	-2)	/2				
	g_6	(0	1	3	14	17	16	9	2)	/2				
	g_7	(0	1	1	10	13	10	2	0)	/2				
Family F14:0b (transpose of F14:0a, conjugate of F14:6b)														
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 2a+1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2a & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2a & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 2a & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 2a & 1 \\ 0 & -1 & 0 & -1 & -1 & -1 & 2a \end{pmatrix}$	Order	(1	2	15	20	21	12	4	0)	/2				
	g_1	(0	3	0	26	27	25	10	0)	/2				
	g_2	(0	0	1	-14	-12	-22	-4	0)	/2				
	g_3	(0	0	6	9	23	21	10	0)	/2				
	g_4	(0	0	4	1	-6	-5	2	0)	/2				
	g_5	(0	1	3	-2	-7	-7	-9	-2)	/2				
	g_6	(0	2	5	13	13	14	3	2)	/2				
	g_7	(0	2	7	43	38	30	4	0)	/2				

Table 8.14: Degree 15, diameter class 0, $a = 2k/7$

LGM							Polynomial in $2a$							
Family F15:0 (self-transpose, self-conjugate, translate of F14:3)														
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order	(1	0	14	0	21	0	7	0)					
	g_1	(0	1	4	10	2	9	-2	1)					
	g_2	(0	1	4	4	12	3	6	-1)					
	g_3	(0	1	2	0	4	1	4	1)					
	g_4	(0	1	0	-2	0	-7	0	-1)					
	g_5	(0	1	-2	0	-4	1	-4	1)					
	g_6	(0	1	-4	4	-12	3	-6	-1)					
	g_7	(0	1	-4	10	-2	9	2	1)					

8.5 Dimension 8, degrees 16 and 17

As the dimension is even, there is no translation between graph families of degree 16 and 17. Also, as the dimension is even, the graph families are defined for diameter classes modulo $f/2 = 4$. So there are four classes of largest-known circulant graph families for each degree.

For degree 16, for all four diameter classes there are transpose pairs of non-isomorphic families. They form two sets of conjugate pairs. For degree 17, for all four diameter classes there are transpose pairs of non-isomorphic families. Two of the classes form a set of conjugate pairs, see Table 8.15.

These relationships are summarised in the dimensional frame, Table 8.16.

Table 8.15: Order of largest-known circulant graph families of degrees 16 and 17

Degree 16 families		Degree 17 families	
F16:	Order polynomial in $2a$	F17: Order polynomial in $2a$	c^*
0a/b	$(1\ 2\ 20\ 28\ 11\ 2\ -4\ -4\ 0)/2$	0a/b $(1\ 0\ 17\ 0\ 4\ 6\ -4\ 0\ 0)$	0
1a/b	$(1\ 6\ 33\ 100\ 183\ 212\ 151\ 60\ 10)/2$	1a/b $(1\ 4\ 25\ 60\ 82\ 78\ 50\ 20\ 4)$	-1
2a/b	$(1\ -6\ 33\ -100\ 183\ -212\ 151\ -60\ 10)/2$	2a/b $(1\ -8\ 44\ -150\ 320\ -436\ 372\ -184\ 40)$	2
3a/b	$(1\ -2\ 20\ -28\ 11\ -2\ -4\ 4\ 0)/2$	3a/b $(1\ -4\ 25\ -60\ 82\ -78\ 50\ -20\ 4)$	1

* for degree 16, $a = (k + c)/4$ * for degree 17, $a = (k + c)/4$

Table 8.16: Dimension 8: largest-known circulant graph families

Degree 16 families		Degree 17 families	
		F17:2a/b	
F16:0a/b		F17:3a/b	
↕	F16:1a/b	↕	F17:0a/b
	F16:2a/b		F17:1a/b
F16:3a/b			

For diameter class 0, the LGMs and formulae for degrees 16 and 17 are shown in Tables 8.17 and 8.18, with a full set in Appendix A.7. Properties of the individual graphs are given in Appendix D.

Table 8.17: Degree 16, diameter class 0, $a = k/4$

LGM odd basis								Polynomial in $2a$										
Family F16:0a (transpose of F16:0b, conjugate of F16:3a)																		
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 2a & 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2a & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 2a & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2a & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 & -1 & 2a-1 \end{pmatrix}$								Order	(1	2	20	28	11	2	-4	-4	0)/2
								$g1$	(0	1	3	22	14	-3	-7	-6	0)/2
								$g2$	(0	0	2	3	5	-5	-7	-2	0)/2
								$g3$	(0	0	1	-6	-4	-8	-11	0	0)/2
								$g4$	(0	0	1	10	18	12	7	2	-2)/2
								$g5$	(0	1	1	10	10	-1	-5	-6	-2)/2
								$g6$	(0	0	0	5	25	27	5	-2	0)/2
								$g7$	(0	1	0	15	15	-14	-13	0	0)/2
								$g8$	(0	1	4	19	37	36	19	4	0)/2
Family F16:0b (transpose of F16:0a, conjugate of F16:3b)																		
$\begin{pmatrix} 2a+1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 2a+1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 2a+1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 2a & 0 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 2a & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 2a & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 2a & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$								Order	(1	2	20	28	11	2	-4	-4	0)/2
								$g1$	(0	2	0	34	21	-8	-3	-2	0)/2
								$g2$	(0	0	2	4	-3	12	3	-6	0)/2
								$g3$	(0	0	1	11	12	11	1	-8	0)/2
								$g4$	(0	0	1	-6	-22	-14	3	4	2)/2
								$g5$	(0	1	3	14	6	-3	5	0	-2)/2
								$g6$	(0	0	1	-6	22	40	-3	-14	0)/2
								$g7$	(0	2	6	45	63	9	-21	-8	0)/2
								$g8$	(0	1	2	10	15	3	-7	-4	0)/2

Table 8.18: Degree 17, diameter class 0, $a = k/4$

LGM								Polynomial in $2a$									
Family F17:0a (transpose of F17:0b)																	
$\begin{pmatrix} 2a+1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 2a+1 & -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & a2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & a2 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 2a & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 2a & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 2a-1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & -1 & 2a-1 \end{pmatrix}$								Order	(1	0	17	0	4	6	-4	0	0)
								$g1$	(0	1	-1	16	-15	1	-1	-10	1)
								$g2$	(0	1	-1	17	-13	1	5	-11	1)
								$g3$	(0	1	-1	18	-18	8	10	-11	1)
								$g4$	(0	1	0	18	-1	7	0	-10	1)
								$g5$	(0	1	0	16	-3	2	0	-9	1)
								$g6$	(0	1	1	17	14	2	14	8	-1)
								$g7$	(0	0	1	0	15	12	13	8	-1)
								$g8$	(0	1	1	17	21	9	15	9	-1)
Family F17:0b (transpose of F17:0a)																	
$\begin{pmatrix} 2a+1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 2a+1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & a2 & 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & a2 & -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 2a & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 2a & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & -1 & 2a-1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$								Order	(1	0	17	0	4	6	-4	0	0)
								$g1$	(0	1	-4	15	-13	10	4	-6	1)
								$g2$	(0	1	-1	3	-2	-1	4	-5	1)
								$g3$	(0	1	2	22	-6	-12	7	3	-1)
								$g4$	(0	0	4	4	18	24	-5	-6	1)
								$g5$	(0	1	3	8	-1	4	-5	-3	1)
								$g6$	(0	1	1	4	23	-9	-9	6	-1)
								$g7$	(0	1	-2	20	17	3	-4	-4	1)
								$g8$	(0	1	1	3	4	1	4	3	-1)

8.6 Dimension 9, degrees 18 and 19

As the dimension is odd, there is translation between degree 18 graph families of odd order and degree 19 families. Also, as the dimension is odd, the graph families are defined for diameter classes modulo $f = 9$, so there are nine classes of largest-known circulant graph families for each degree.

For degree 18, for eight of the nine diameter classes there are transpose pairs of non-isomorphic families. Six of these form conjugate pairs. Only one of these pairs has odd order and so translates to circulant graph families of degree 19. The final diameter class is self-transpose, self-conjugate and also translates to a degree 19 circulant graph family. For three of the initial eight diameter classes, classes 1, 3 and 5, the graph families are circulant only for some of the diameters in their class. For the other diameters, the order and generators all share a common factor in a regular cycle, resulting in graphs that are Abelian Cayley with cyclic rank 2 (instead of 1 for circulants). For these diameters, different graph families, F18:1c/d, F18:3 c/d and F18:5c/d, provide the largest-known circulant graphs. It happens that F18:1c/d are conjugates of F18:7a/b.

For degree 19, there are transpose pairs of non-isomorphic families for six of the nine diameter classes. Only one of these pairs translates to largest-known degree 18 families. There is also a conjugate pair of self-transpose families, and a single

diameter class that is self-transpose, self-conjugate and translates to a largest-known degree 18 family, see Table 8.19.

Table 8.19: Order of largest-known circulant graph families of degrees 18 and 19

Degree 18 families F18: Order polynomial in $2a$	Degree 19 families F19: Order polynomial in $2a$	c^*
0ab (1 2 23 34 59 52 35 16 4 0)/2	5 (1 2 22 32 62 60 47 26 9 2)	0
1ab (1 6 37 126 265 346 267 112 20 0)/2	6ab (1 6 36 120 253 350 317 184 63 10)	-2
1cd (1 6 37 122 251 342 305 172 56 8)/2		-2
2ab (1 -8 50 -194 462 -698 672 -394 125 -16)/2	7ab (1 -8 50 -194 462 -698 672 -394 125 -16)	5
3ab (1 -4 29 -74 115 -122 81 -34 8 0)/2	8ab (1 -4 27 -66 109 -126 104 -62 23 -4)	3
3cd (1 -4 28 -70 118 -132 96 -42 8 0)/2		3
4 (1 0 20 0 58 0 43 0 9 0)/2	0 (1 0 20 0 58 0 43 0 9 0)	1
5ab (1 4 29 74 115 122 81 34 8 0)/2	1ab (1 4 27 68 122 146 119 66 23 4)	-1
5cd (1 4 28 72 117 122 89 42 8 0)/2		-1
6ab (1 8 50 194 462 698 672 394 125 16)/2	2ab (1 8 50 194 462 698 672 394 125 16)	-3
7ab (1 -6 37 -122 251 -342 305 -172 56 -8)/2	3ab (1 -6 36 -118 245 -338 313 -190 67 -10)	4
8ab (1 -2 23 -34 59 -52 35 -16 4 0)/2	4 (1 -2 22 -32 62 -60 47 -26 9 -2)	2
* for degree 18, $a = (2k + c)/9$		
* for degree 19, $a = (2k + c - 1)/9$		

These relationships are summarised in the dimensional frame, Table 8.20.

Table 8.20: Dimension 9: largest-known circulant graph families

Degree 18 families	Degree 19 families
F18:0a/b	F19:5
F18:1a/b*	F19:6a/b
F18:2a/b	F19:7a/b
F18:3a/b*	F19:8a/b
F18:4	F19:0
F18:5a/b*	F19:1a/b
F18:6a/b	F19:2a/b
F18:7a/b	F19:3a/b
F18:8a/b	F19:4

* Cyclic rank 2 for some diameters.

For diameter class 0, the LGMs and formulae for degrees 18 and 19 are shown in Tables 8.21 and 8.22, with a full set in Appendix A.8. Properties of the individual graphs are given in Appendix D.

Table 8.21: Degree 18, diameter class 0, $a = 2k/9$

LGM odd basis									Polynomial in $2a$											
Family F18:0a (transpose of F18:0b, conjugate of F18:8a)																				
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 2a & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2a & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order	(1	2	23	34	59	52	35	16	4	0)	/2								
	$g1$	(0	0	1	7	16	11	12	5	0	0)	/2								
	$g2$	(0	0	1	2	-6	-5	-8	-4	-2	0)	/2								
	$g3$	(0	0	2	4	12	9	10	9	4	0)	/2								
	$g4$	(0	0	1	4	3	8	11	3	0	0)	/2								
	$g5$	(0	0	1	-5	-8	-24	-24	-19	-9	-2)	/2								
	$g6$	(0	1	3	18	26	35	28	16	7	2)	/2								
	$g7$	(0	0	1	-3	6	11	11	8	2	0)	/2								
	$g8$	(0	0	1	7	19	24	15	6	2	0)	/2								
$g9$	(0	1	1	16	20	23	21	10	2	0)	/2									
Family F18:0b (transpose of F18:0a, conjugate of F18:8b)																				
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 2a & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2a & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 2a & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 2a & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & 2a & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 2a \end{pmatrix}$	Order	(1	2	23	34	59	52	35	16	4	0)	/2								
	$g1$	(0	1	2	15	15	24	15	7	2	0)	/2								
	$g2$	(0	0	2	1	6	4	2	4	0	0)	/2								
	$g3$	(0	0	2	4	5	2	-2	-5	-2	0)	/2								
	$g4$	(0	0	2	2	11	12	15	7	2	0)	/2								
	$g5$	(0	0	1	10	16	27	25	18	9	2)	/2								
	$g6$	(0	1	1	13	18	32	27	17	7	2)	/2								
	$g7$	(0	0	0	7	5	14	10	4	0	0)	/2								
	$g8$	(0	0	1	-4	-2	-9	-1	2	0	0)	/2								
$g9$	(0	0	1	-2	-8	-8	-11	-6	0	0)	/2									

Table 8.22: Degree 19, diameter class 0, $a = 2k/9$

LGM									Polynomial in $2a$									
Family F19:0 (self-transpose, self-conjugate, translate of F18:4)																		
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 & 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 2a & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2a & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2a & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 & 2a & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 & 2a & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 2a \end{pmatrix}$	Order	(1	0	20	0	58	0	43	0	9	0)							
	$g1$	(0	1	4	15	45	30	57	13	17	1)							
	$g2$	(0	1	-2	11	-25	22	-23	9	-3	1)							
	$g3$	(0	1	0	8	1	18	-6	8	-3	-1)							
	$g4$	(0	1	-1	14	9	9	3	-4	-1	-1)							
	$g5$	(0	1	-2	12	-13	8	-4	4	-1	1)							
	$g6$	(0	1	0	26	-4	40	-16	17	-6	1)							
	$g7$	(0	0	2	-6	-1	-11	-20	-3	-10	1)							
	$g8$	(0	0	2	-4	5	-33	8	-17	4	-1)							
	$g9$	(0	1	1	20	-13	37	-11	10	-3	-1)							

8.7 Dimension 10, degree 20

As the dimension is even, there is no translation between degree 20 graph families of odd order and degree 21 families. The investigation was not progressed beyond degree 20, as the computer runs for proper analysis began to exceed the available resources. As the dimension is even, the graph families are defined for diameter classes modulo $f/2 = 5$, so there are five classes of largest-known circulant graph families for degree 20.

For degree 20, for all five diameter classes there are transpose pairs of non-isomorphic families. Two of these, classes 1 and 3, form conjugate pairs. For these two classes, the graph families F20:1a/b and F20:3a/b are circulant only for some of the diameters and are Abelian Cayley graph families of cyclic rank 2 for the others. This is similar to three of the diameter classes of degree 18. For these two diameter classes, different graph families, F20:1c/d and F20:3c/d, provide the largest-known circulant graphs, also transpose pairs in a conjugate pair, see Table 8.23.

Table 8.23: Order of largest-known circulant graph families of degree 20

Degree 20 families		
F20:	Order polynomial in $2a$	c^*
0a/b	(1 2 26 42 93 92 86 46 16 4 0)/2	0
1a/b	(1 6 42 150 337 512 526 352 142 28 0)/2	-1
1c/d	(1 6 41 144 325 500 535 398 198 60 8)/2	-1
2a/b	(1 10 70 322 976 1996 2776 2584 1533 518 74)/2	-2
3a/b	(1 -6 42 -150 337 -512 526 -352 142 -28 0)/2	2
3c/d	(1 -6 41 -144 325 -500 535 -398 198 -60 8)/2	2
4a/b	(1 -2 26 -40 89 -92 77 -44 18 -4 0)/2	1

* for degree 20, $a = (k + c)/5$

These relationships are summarised in the dimensional frame, Table 8.24.

Table 8.24: Dimension 10: largest-known circulant graph families

Degree 20 families		(Degree 21 not investigated)
F20:0a/b		
	F20:1a/b*	
	\updownarrow	F20:2a/b
	F20:3a/b*	
F20:4a/b		

* Cyclic rank 2 for some diameters.

For diameter class 0, the LGM odd bases and formulae for degree 20 are shown in Table 8.25, with a full set in Appendix A.9. Properties of the individual graphs are given in Appendix D.

Table 8.25: Degree 20, diameter class 0, $a = k/5$

Family F20:0a (transpose of F20:0b)											
LGM odd basis											
	$2a+1$	0	-1	-1	-1	-1	-1	0	0	0	
	0	$2a+1$	0	0	0	-1	-1	-1	0	0	
	1	0	$2a$	0	0	-1	-1	-1	-1	-1	
	1	0	0	$2a$	-1	-1	-1	-1	-1	0	
	1	0	0	1	$2a$	0	0	0	-1	-1	
	1	1	1	1	0	$2a$	0	0	-1	-1	
	1	1	1	1	0	0	$2a$	0	-1	-1	
	0	1	1	1	0	0	0	$2a$	-1	0	
	0	0	1	1	1	1	1	1	$2a$	0	
	0	0	1	0	1	1	1	0	0	$2a$	
Polynomial in $2a$											
Order	(1	2	26	42	93	92	86	46	16	4	0)/2
$g1$	(0	1	0	17	11	42	19	26	4	2	0)/2
$g2$	(0	0	1	3	3	5	6	2	4	-2	0)/2
$g3$	(0	0	2	2	7	1	1	1	0	4	0)/2
$g4$	(0	0	2	4	10	8	14	6	6	2	0)/2
$g5$	(0	0	1	-4	-3	-17	-18	-21	-12	-4	0)/2
$g6$	(0	0	1	-7	-8	-27	-24	-28	-15	-8	-2)/2
$g7$	(0	1	3	19	34	66	68	58	31	8	2)/2
$g8$	(0	0	0	6	18	28	36	24	14	2	0)/2
$g9$	(0	0	1	9	8	22	3	5	-8	-4	0)/2
$g10$	(0	0	1	7	10	30	23	27	10	4	0)/2
Family F20:0b (transpose of F20:0a)											
LGM odd basis											
	$2a+1$	0	1	1	1	1	1	1	0	0	0
	0	$2a+1$	0	0	0	1	1	1	1	0	0
	-1	0	$2a$	0	0	1	1	1	1	1	1
	-1	0	0	$2a$	1	1	1	1	1	1	0
	-1	0	0	-1	$2a$	0	0	0	0	1	1
	-1	-1	-1	-1	0	$2a$	0	0	0	1	1
	-1	-1	-1	-1	0	0	$2a$	0	0	1	1
	0	-1	-1	-1	0	0	0	$2a$	1	0	0
	0	0	-1	-1	-1	-1	-1	-1	$2a$	0	0
	0	0	-1	0	-1	-1	-1	0	0	$2a$	0
Polynomial in $2a$											
Order	(1	2	26	42	93	92	86	46	16	4	0)/2
$g1$	(0	1	2	19	19	48	31	30	10	2	0)/2
$g2$	(0	0	1	-1	3	7	4	12	2	2	0)/2
$g3$	(0	0	2	4	13	17	17	11	6	0	0)/2
$g4$	(0	0	2	2	10	6	16	6	4	2	0)/2
$g5$	(0	0	1	6	13	17	12	11	-2	0	0)/2
$g6$	(0	0	1	9	16	33	36	32	21	8	2)/2
$g7$	(0	1	1	17	26	60	56	54	25	8	2)/2
$g8$	(0	0	0	6	0	16	4	8	0	-2	0)/2
$g9$	(0	0	1	-5	-10	-26	-29	-29	-14	-4	0)/2
$g10$	(0	0	1	-3	-2	-16	-7	-9	0	0	0)/2

BIPARTITE CIRCULANT GRAPH FAMILIES

In Chapter 9, largest-known bipartite circulant graph families up to degree 11 are presented, along with some theorems establishing how bipartite circulant graph families of any degree are related to corresponding non-bipartite families. The bipartite/non-bipartite relationship, alongside the three previously discussed relationships of translation, conjugation and transposition are presented graphically for each dimension by a dimensional frame. An extremal order conjecture for bipartite circulant graph families is presented.

A bipartite graph is a graph whose vertices can be divided into two disjoint sets such that every edge connects a vertex from one set to a vertex of the other. Clearly, such a graph does not contain any odd length cycles. To avoid ambiguity, in this chapter we will use the term *general* circulant graph to distinguish undirected circulant graphs that are not restricted to be bipartite. For each degree d and diameter k , we denote by $Ext_{circ}^{bip}(d, k)$ the order of an extremal bipartite circulant graph and by $Ext_{circ}(d, k)$ the order of an extremal general circulant graph. Similarly, $LK_{circ}^{bip}(d, k)$ and $LK_{circ}(d, k)$ denote corresponding largest-known graphs.

As any Cayley graph is vertex transitive the order of the two partite sets must be equal and therefore the order, n , of any bipartite circulant graph is even. When the degree is odd, then the connection set will include the unique involutory element $n/2$, creating an edge between vertices 0 and $n/2$, which must therefore lie in different partite sets. Thus, for odd degree we have the additional constraint on the order, that $n \equiv 2 \pmod{4}$. In case the generating set is primitive, so that the odd vertices comprise one partite set and the even vertices the other, then all the generator elements must be odd.

9.1 Relations between bipartite circulant graphs and general circulant graphs

In this section, we discuss two important relations between bipartite circulant graphs and certain categories of general circulant graphs of the same dimension. The first is a mapping from a category of general circulant graphs of odd degree to bipartite circulant graphs of even degree, Theorem 9.1.

Theorem 9.1. *Let X be a circulant graph of odd degree $d = 2f + 1$ and order $n = 2m$ where m is even, with generating set $\{g_1, \dots, g_f\}$ where $g_i < m$ is odd for $i = 1, \dots, f$, and diameter k . Then there exists a bipartite circulant graph X' of degree $d' = 2f$ and order $n' = m$ with generating set $\{g'_1, \dots, g'_f\}$ where $g'_i = g_i$ if $g_i \leq m/2$ and $g'_i = m - g_i$ otherwise, and with diameter $k' = k - 1$ or k . If X contains a vertex v such that the distance $d_X(0, v) = d_X(0, v + m) = k$, then $k' = k$; otherwise $k' = k - 1$.*

Proof. As graph X has odd degree, its connection set includes the involutory element m . The graph X' is created from X by identifying involutory pairs of vertices $\{i, i + m\}$ for $i = 0, \dots, m - 1$, so that i and $i + m$ in X are both mapped to i in X' . This also maps the generating set $\{g_1, \dots, g_f\}$ to $\{g'_1, \dots, g'_f\}$ as required. The involution for each vertex is eliminated and the degree of X' is reduced to $2f$. As the order of X' is even and all the elements of its connection set are odd, the graph is bipartite.

Let v be a vertex of X' , so that v and $v + m$ are vertices of X . As X has diameter k we have distance $d_X(0, v) \leq k$ and $d_X(0, v + m) \leq k$. If the shortest path in X from 0 to v includes an edge defined by the involution m , then $d_X(0, v + m) \leq k - 1$ with a path excluding m . Otherwise, the shortest path in X from 0 to v excludes the edge m . In either case, the path is identified with a path in X' from 0 to v of length $l \leq k$. Hence, the diameter of X' , $k' \leq k$. If there exists a vertex v in X such that $d_X(0, v) = d_X(0, v + m) = k$ then $d_{X'}(0, v) = k$, so that $k' = k$; otherwise $k' = k - 1$. \square

The second relation is a 1-1 mapping between a category of general circulant graphs of even degree and bipartite circulant graphs of odd degree, Theorems 9.2 and 9.3.

Theorem 9.2. *Let X be a bipartite circulant graph of odd degree $d = 2f + 1$, order $n = 2m$ where m is odd, and diameter k , with generating set $\{g_1, \dots, g_f\}$ where $g_i < m$ is necessarily odd for $i = 1, \dots, f$. Then there exists a circulant graph X' of degree $d' = 2f$, order $n' = m$ and diameter $k' = k - 1$ with generating set $\{g'_1, \dots, g'_f\}$ where $g'_i = g_i$ if $g_i \leq m/2$ and $g'_i = |g_i - m|$ otherwise.*

Proof. As graph X has odd degree, its connection set includes the involutory element m . The graph X' is created from X by identifying involutory pairs of vertices $\{i, i + m\}$ for $i = 0, \dots, m - 1$, so that i and $i + m$ in X are both mapped to i in X' . This also maps the generating set $\{g_1, \dots, g_f\}$ to $\{g'_1, \dots, g'_f\}$ as required. The involution for each vertex is eliminated and the degree of X' is reduced to $2f$. Let v be a vertex of X' , so that v and $v + m$ are vertices of X . As X has diameter k , we

have distance $d_X(0, v) \leq k$ and $d_X(0, v + m) \leq k$. If $d_X(0, v) = k$, then we cannot also have $d_X(0, v + m) = k$ as this would create a closed path of length $2k + 1$, contrary to the assumption that X is bipartite. Hence, either the distance $d_X(0, v) \leq k - 1$ or $d_X(0, v + m) \leq k - 1$, so that the diameter of X' , $k' = k - 1$. \square

Theorem 9.3. *Let X' be a circulant graph of even degree $d' = 2f$, order $n' = m$ where m is odd, and diameter $k' = k - 1$, with generating set $\{g'_1, \dots, g'_f\}$ where $g'_i < m/2$. Then there exists a bipartite circulant graph X of degree $d = 2f + 1$, order $n = 2m$ and diameter k , with generating set $\{g_1, \dots, g_f\}$ where $g_i = g'_i$ if g'_i is odd and $g_i = |g'_i - m|$ otherwise.*

Proof. We create graph X from X' by lifting each vertex v of X' to an adjacent pair of vertices v and $v + m$ in X , connected by the addition of the involution m to the connection set. The generating set of X , $\{g_1, \dots, g_f\}$ is defined by $g_i = g'_i$ if g'_i is odd, and $g_i = |g'_i - m|$ otherwise, so that the new generators are all odd. Hence the graph X is bipartite. Let v be a vertex of X' , so that v and $v + m$ are adjacent vertices of X . Then there is a shortest path in X' from 0 to v of length $l \leq k' = k - 1$, say $(\delta_1 g'_{i_1}, \dots, \delta_l g'_{i_l})$ where $\delta_i = \pm 1$. Thus, $(\delta_1 g_{i_1}, \dots, \delta_l g_{i_l})$ defines a shortest path in X from 0 to v or $v + m$ of length l and excluding any edge defined by the involution m . Therefore, the distance from 0 to the other vertex of the pair in X is $l + 1$. Hence X has diameter k . \square

These theorems are particularly useful in the consideration of extremal and largest-known circulant graphs because most of the families of extremal and largest-known general circulant graphs belong to the categories covered by the theorems. Moreover, it emerges that each family of extremal and largest-known bipartite circulant graphs is closely related, in terms of their order and generating sets, with one or more families of extremal and largest-known general circulant graphs of the same dimension.

Within any given odd dimension, if we consider odd degree families of general circulant graphs, then Theorem 9.1 and Theorem 5.8 define mappings from this domain to even-degree circulant graphs that are respectively bipartite and odd-order general. This establishes a direct relation between bipartite circulant graphs and odd-order general circulant graphs of the same even degree and odd dimension. It emerges that this relation extends also to general circulant graphs of even order, despite the non-existence of corresponding odd degree graphs. This is formalised in Theorem 9.4.

Theorem 9.4. *For any odd dimension f and any $k_{\mathcal{X}}$ where $0 \leq k_{\mathcal{X}} < f$, let $\mathcal{X}(2f, k_{\mathcal{X}})$ be a family of bipartite circulant graphs $X(2f, k)$ of even degree $2f$ and diameter k for any $k \equiv k_{\mathcal{X}} \pmod{f}$, with order defined by a polynomial $n_{\mathcal{X}}(a)$ of degree f in the parameter $a = (4k + c)/f$ where*

$$c = \begin{cases} (-4k_{\mathcal{X}}) \pmod{f} & \text{for } k_{\mathcal{X}} < (f-1)/2 \\ 2 & \text{for } k_{\mathcal{X}} = (f-1)/2 \\ 4 - \{4(k_{\mathcal{X}} + 1) \pmod{f}\} & \text{for } k_{\mathcal{X}} > (f-1)/2, \end{cases}$$

c being chosen so that a is integral, and with generating set $G = \{g_1(a), \dots, g_f(a)\}$ where $g_i(a)$ are polynomials of degree at most f and taken $\pmod{n_{\mathcal{X}}(a)}$.

For $k_{\mathcal{Y}} \equiv (k_{\mathcal{X}} + (f-1)/2) \pmod{f}$, we define $\mathcal{Y}(2f, k_{\mathcal{Y}})$ to be the family of general circulant graphs $Y(2f, k')$ of the same even degree $2f$ for any $k' \equiv k_{\mathcal{Y}} \pmod{f}$ with order $n_{\mathcal{Y}}(a') = n_{\mathcal{X}}(a')$ where $a' = (4k' + c + 2)/f$ and with generating set $G' = \{g_1(a'), \dots, g_f(a')\} \pmod{n_{\mathcal{Y}}(a')}$. Then for any such k' , $Y(2f, k')$ has diameter k' . The converse also holds.

The extremal and largest-known general circulant graph families of odd degrees 3 to 19, and all diameter classes, have odd generating sets (sets where all generators are odd). Therefore, Theorem 9.1 can be applied, and the resultant even-degree families are extremal and largest-known bipartite circulant graph families for even degrees 2 to 18, respectively, for all diameter classes. There is one exception, for the degree 8 odd-diameter bipartite graph family. In this case, there is a degree 9 graph family for even diameter k with the property that its graphs do not contain any vertex v such that the distance $d(0, v) = d(0, v + m) = k$, where $2m$ is the order of the graph. Thus, by the final part of the Theorem 9.1, the resultant family is the largest-known bipartite circulant graph family of degree 8 and odd diameter class.

Similarly, by Theorem 9.3, the extremal and largest-known odd-degree bipartite circulant graph families are derived from the extremal and largest-known even-degree general circulant graph families of odd order. The extremal families of degree 2 and 4, and the largest-known families of degree 6, have odd order for all diameter classes, and so Theorem 9.3 may be applied. For degree 8, the largest-known odd-order families are the ones found by Monakhova [38], and Theorem 9.3 is applied to them. For higher even degrees, the order of the largest-known general circulant graphs may be odd or even depending on the diameter class. In each case, it is the largest-known family of odd order that is used to construct the largest-known bipartite circulant graph family for each diameter class for each odd degree.

The relationships between general circulant graph families and bipartite graph families defined by Theorems 9.1, 9.2 and 9.3 are reflected by similar relationships between their LGMs and mappings such as transposition, conjugation and translation. This is demonstrated with the example of dimension 3, degrees 6 and 7. In Table 9.1, the general circulant dimensional frame from Table 8.4 is shown, followed by the corresponding dimensional frame for largest-known bipartite circulants. Then in the same format, the LGMs and order polynomials common to

Table 9.1: Comparison between largest-known general and bipartite circulant graph families of dimension 3

Largest-known general circulant graph families						
$a =$	Degree 6 families		Degree 7 families			$a =$
$2k/3$	F6:0A	\longleftrightarrow			F7:2A	$(2k-1)/3$
$2k/3$	\uparrow	F6:0B	\longleftrightarrow		F7:2B	$(2k-1)/3$
$(2k+1)/3$	\downarrow	\updownarrow	F6:1	\longleftrightarrow	F7:0	$2k/3$
$(2k+2)/3$		F6:2B	\longleftrightarrow		F7:1B	$(2k+1)/3$
$(2k+2)/3$	F6:2A	\longleftrightarrow			F7:1A	$(2k+1)/3$
<hr/>						
Largest-known bipartite circulant graph families						
$a =$	Degree 6 families		Degree 7 families			$a =$
$(2k-1)/3$	D6:2A	\longleftrightarrow			D7:1A	$(2k-2)/3$
$(2k-1)/3$	\uparrow	D6:2B	\longleftrightarrow		D7:1B	$(2k-2)/3$
$2k/3$	\downarrow	\updownarrow	D6:0	\longleftrightarrow	D7:2	$(2k-1)/3$
$(2k+1)/3$		D6:1B	\longleftrightarrow		D7:0B	$2k/3$
$(2k+1)/3$	D6:1A	\longleftrightarrow			D7:0A	$2k/3$
<hr/>						
Largest-known general and bipartite circulant graph families						
Degree 6 LGM odd bases			Degree 7 LGMs			
$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$		\longleftrightarrow			$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$	
\uparrow	$\begin{pmatrix} 2a+2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	\longleftrightarrow		$\begin{pmatrix} 2a+2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	\uparrow	
	\updownarrow	$\begin{pmatrix} 2a-1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	\longleftrightarrow	$\begin{pmatrix} 2a-1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	\downarrow	
	\downarrow	$\begin{pmatrix} 2a-2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	\longleftrightarrow		$\begin{pmatrix} 2a-2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	\downarrow
$\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$		\longleftrightarrow			$\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	
<hr/>						
Largest-known general and bipartite circulant graph families						
Degree 6 order polynomial in $2a$			Degree 7 order polynomial in $2a$			
$(1\ 2\ 3\ 2)/2$		\longleftrightarrow			$(1\ 2\ 3\ 2)$	
\uparrow	$(1\ 2\ 3\ 2)/2$	\longleftrightarrow		$(1\ 2\ 3\ 2)$	\uparrow	
	\updownarrow	$(1\ 0\ 3\ 0)/2$	\longleftrightarrow	$(1\ 0\ 3\ 0)$	\updownarrow	
	$(1\ -2\ 3\ -2)/2$	\longleftrightarrow		$(1\ -2\ 3\ -2)$	\downarrow	
$(1\ -2\ 3\ -2)/2$		\longleftrightarrow			$(1\ -2\ 3\ -2)$	

both frames are presented, where the value of a in each case is different for general and bipartite families as indicated in the first two frames.

It has already been mentioned in Section 8.2 that one LGM can be associated with up to eight distinct isomorphism classes of Abelian Cayley graph families. As a consequence of the relationships described here between general and bipartite graph families, we can see that up to eight further graph families may also be based on the same LGM. This means that up to 16 distinct Abelian Cayley graph families may share a common LGM basis, depending on four defined axes of relationship: transposition, conjugation, translation and bipartition. It should be noted that whilst the first three relationships preserve quasimaximality, bipartite graph families are subquasimaximal, having a lower second coefficient in their order polynomials.

9.2 Dimension 1, degrees 2 and 3

As the dimension is odd, there is translation between graph families of degree 2 and degree 3. Also, as the dimension is odd, the graph families are defined for diameter classes modulo $f = 1$, so there is just one diameter class of extremal bipartite circulant graph families for each degree.

For degree 2, the solutions are trivial, with a single generator element of 1, giving the cycle graph of order $Ext_{circ}^{bip}(2, k) = 2k$. This compares with the solution for general circulant graphs, which are cycle graphs of order $Ext_{circ}(2, k) = 2k + 1$.

For degree 3, the solutions are again trivial, with a single generator element of 1, giving a graph of order $Ext_{circ}^{bip}(3, k) = 4k - 2$ that is a cycle graph with the ‘diameters’ added. This compares with the solution for general circulant graphs, which are similarly shaped graphs, but of order $Ext_{circ}(3, k) = 4k$.

These families are summarised in Table 9.2.

Table 9.2: Order of extremal bipartite circulant graph families of degrees 2 & 3

Degree 2 families		Degree 3 families	
Family	Order polynomial in $2a$	Family	Order polynomial in $2a$
D2	$(1 \ 0)/2$	D3	$(1 \ 0)$
for degree 2, $a = 2k$		for degree 3, $a = (2k - 1)$	

Properties of the individual graphs up to diameter 16 are given in Appendix E, Tables E.1 and E.2.

9.3 Dimension 2, degrees 4 and 5

As the dimension is even, there is no translation between graph families of degree 4 and 5. Also, as the dimension is even, the graph families are defined for diameter classes modulo $f/2 = 1$, so there is just one diameter class of extremal bipartite circulant graph families for each degree.

For degree 4, despite there being only one diameter class, there are multiple isomorphism classes of largest-known bipartite circulant graphs, depending on the diameter k . This was discovered by Tzvieli in 1991 [49], see Theorem 9.5. These graphs are shown in Table 9.3 for diameter $k \leq 12$. Also, see Appendix E, Table E.3.

Table 9.3: Extremal bipartite circulant graphs of degree 4, up to diameter 12

Diameter k	Order $Ext_{circ}^{bip}(4, k)$	Generating sets existing for each parameter t					
		$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	
2	8	1, 3					
3	18	1, 5	1, 7				
4	32	1, 7	1, 9				
5	50	1, 9	1, 11	1, 19	1, 21		
6	72	1, 11	1, 13				
7	98	1, 13	1, 15	1, 27	1, 29	1, 41	1, 43
8	128	1, 15	1, 17			1, 47	1, 49
9	162	1, 17	1, 19	1, 35	1, 37		1, 71
10	200	1, 19	1, 21			1, 59	1, 61
11	242	1, 21	1, 23	1, 43	1, 45	1, 65	1, 67
12	288	1, 23	1, 25				

The graphs have order $Ext_{circ}^{bip}(4, k) = 2k^2$, a quadratic in k reflecting the two degrees of freedom in specifying the generating set. This compares with the extremal solution for general circulant graphs, which has order $Ext_{circ}(4, k) = 2k^2 + 2k + 1$.

Theorem 9.5. (Tzvieli [49]) *Given any k , then for any t with $1 \leq t \leq \lfloor (k-1)/2 \rfloor$ such that $\gcd(t, k) = 1$, there exists a bipartite circulant graph of degree 4, diameter k and order $2k^2$ with generating sets $\{1, 2tk - 1\}$ and $\{1, 2tk + 1\}$. For each t , the pair of generating sets create isomorphic graphs that belong to a distinct isomorphism class.*

It is easily observed that the two generating sets $\{1, 2tk - 1\}$ and $\{1, 2tk + 1\}$ generate isomorphic graphs by multiplying the first by $2tk + 1$ to achieve the second.

Therefore, for any bipartite circulant graph of degree 4, diameter k and order $2k^2$, the number of distinct generating sets is equal to the Euler totient function $\phi(k)$, being a count of the numbers below k that are coprime with k . Also, for $k > 2$ the number of distinct isomorphism classes is equal to $\phi(k)/2$.

The obvious question is why there are so many distinct isomorphism classes of degree 4 bipartite circulant graph families, increasing without limit with diameter. Is it possible that there can be so many distinct LGM odd bases within the format of a 2×2 matrix? In fact, the answer is yes.

The canonical LGM odd basis that reflects the relationship of Theorem 9.1 is the LGM for the extremal circulant graph family of degree 5. It is $\begin{pmatrix} 2a+1 & -1 \\ 1 & 2a-1 \end{pmatrix}$, and it generates a degree 4 bipartite circulant graph family with generating sets that correspond with parameter value $t = 1$ for each diameter.

Let us consider a generalisation of this LGM: $\begin{pmatrix} 2a+s & -s \\ s & 2a-s \end{pmatrix}$. For $s > 1$, this is not in canonical format, but its determinant has the same value of $4a^2$ for any s . Considered as LGMs of degree 5 Abelian Cayley graphs, for $s > 1$ the resultant families have order polynomial in $2a$ of $n = (1 \ 0 \ 0)$. They are all subquasimaximal because the diameter for each value of a is greater by 1 than the quasimaximal family with $s = 1$, that is $k = a + 1$ instead of $k = a$. These degree 5 subquasimaximal families for $s > 1$ all satisfy the final condition of Theorem 9.1 that there is no vertex v with $d(0, v) = d(0, v + n/2) = k$ where k is the diameter. Therefore, by Theorem 9.1, for $s \geq 1$, $\begin{pmatrix} 2a+s & -s \\ s & 2a-s \end{pmatrix}$ is the LGM odd basis of a bipartite circulant graph family of degree 4 and order $(1 \ 0 \ 0)/2$ for $a = k/2$.

For each value of s , the LGM generates a unique self-transpose, self-conjugate, bipartite circulant graph family, which we denote by D4:s. It is not the case that a family D4:s, for arbitrary s , has a generating set with parameter $t = s$ for any diameter k . There is no simple relationship between the parameters s and t . For any s , the graph family D4:s only contains bipartite circulant graphs for diameters k such that the greatest common divisor $\gcd(s, k) = 1$. Whenever $\gcd(s, k) > 1$, the graph generated by the LGM is a bipartite Abelian Cayley graph of cyclic rank 2 where one of the cyclic orders has the value $\gcd(s, k)$. For $\gcd(s, k) = 1$, it emerges that $t \in \mathbb{N}$ is the lowest number such that $|st - uk| = 1$ for some $u \in \mathbb{N}$. In this way, the parameter value t for any diameter k can be determined for any family D4:s. These values are presented in Table 9.4 for $s \leq 8$ and $k \leq 16$.

For degree 5, there is just one family of extremal bipartite circulant graphs for each diameter k (see Appendix E), Table E.4. Their order is given by $Ext_{circ}^{bip}(5, k) = 4k^2 - 4k + 2$, the quadratic in k again reflecting the two degrees of freedom in specifying the generating set. This compares with the extremal solution for general circulant graphs, which has order $4k^2$. In contrast to the degree 4 case, there is just one generating set for each diameter.

Table 9.4: Value of parameter t for diameter k in graph family D4: s

Family D4: s	Diameter, k														
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s = 1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$s = 2$		1		2		3		4		5		6		7	
$s = 3$	1		1	2		2	3		3	4		4	5		5
$s = 4$		1		1		2		2		3		3		4	
$s = 5$	1	1	1		1	3	3	2		2	5	5	3		3
$s = 6$				1		1				2		2			
$s = 7$	1	1	1	2	1		1	4	3	3	5	2		2	7
$s = 8$		1		2		1		1		4		5		1	

The order of these families are summarised in Table 9.5.

Table 9.5: Order of extremal bipartite circulant graph families of degrees 4 and 5

Degree 4 family		Degree 5 family	
Family	Order polynomial in $2a$	Family	Order polynomial in $2a$
D4: s ($s \geq 1$)	$(1 \ 0 \ 0)/2$	D5	$(1 \ -2 \ 2)$
for degree 4, $a = k$		for degree 5, $a = k$	

The LGMs and formulae for order and a generating set for graph families of all diameter classes are given in Appendix B.1, and properties of the individual graphs up to diameter 16 in Appendix E, Tables E.3 and E.4.

9.4 Dimension 3, degrees 6 and 7

As the dimension is odd, there is translation between graph families of degree 6 and degree 7, and the families are defined for diameter classes modulo $f = 3$, as already shown in the comparison with general circulant graph families of the same dimension, earlier in this chapter.

For degree 6, all the families are self-transpose. For diameter class 0, the family is self-conjugate. For classes 1 and 2, each has two non-isomorphic families forming conjugate pairs.

For degree 7, the same holds with the proviso that the self-conjugate family is in diameter class 2. All these families form translate pairs. This structure exactly mirrors the structure for general circulant graph families of dimension 3, underpinned by sharing common LGMs. The order polynomials are shown in Table 9.6 and the dimensional frame in Table 9.7.

Table 9.6: Order of largest-known bipartite circulant graph families of degrees 6 and 7

Degree 6 families		Degree 7 families		c*
Family	Order polynomial in $2a$	Family	Order polynomial in $2a$	
D6:0	$(1\ 0\ 3\ 0)/2$	D7:2	$(1\ 0\ 3\ 0)$	0
D6:1A/B	$(1\ -2\ 3\ -2)/2$	D7:0A/B	$(1\ -2\ 3\ -2)$	1
D6:2A/B	$(1\ 2\ 3\ 2)/2$	D7:1A/B	$(1\ 2\ 3\ 2)$	-1

* for degree 6, $a = (2k + c)/3$ * for degree 7, $a = (2k + c - 1)/3$

Table 9.7: Dimension 3: Largest-known bipartite circulant graph families

Degree 6 families		Degree 7 families	
D6:2A	\longleftrightarrow	D7:1A	
\uparrow	D6:2B \longleftrightarrow	D7:1B	\uparrow
\downarrow	\updownarrow D6:0 \longleftrightarrow D7:2 \updownarrow	D7:0B	\downarrow
D6:1A	\longleftrightarrow	D7:0A	

The LGMs and formulae for order and generating sets are presented in Appendix B.2, and the properties of the individual graphs up to diameter 16 in Appendix E, Tables E.5 and E.6.

9.5 Dimension 4, degrees 8 and 9

As the dimension is even, there is no translation between graph families of degree 8 and degree 9. Also, the graph families are defined for diameter classes modulo $f/2 = 2$, so there are two diameter classes of largest-known bipartite circulant graph families for each degree.

For degree 8, each diameter class has a self-transpose non-conjugate family. For class 0, family D8:0 is derived from F9:0 by application of Theorem 9.1. However, D8:1 is not derived from F9:1 because there is a larger-order family of diameter class 0 satisfying the final condition of Theorem 9.1. For diameter $k = 4$, the graph of family D8:0 has order 156 and is not extremal. Instead there are two distinct isomorphism classes of graphs, denoted E8:4A and E8:4B, with order 160. Apart from $k = 4$, families D8:0 and D8:1 have been verified to be extremal for diameters $k \leq 7$ and are conjectured to be extremal for all higher diameters.

For degree 9, both diameter classes have a self-transpose family, D9:0 and D9:1, that form a conjugate pair. They are both derived from the largest-known odd-order degree 8 general circulant graph families (discovered by Monakhova [38]) by Theorem 9.3. The order polynomials are presented in Table 9.8.

Table 9.8: Order of largest-known bipartite circulant graph families of degrees 8 and 9

Degree 8 families		Degree 9 families		c^*
Family	Order polynomial in $2a$	Family	Order polynomial in $2a$	
D8:0	$(1\ 0\ 3\ 2\ 0)/2$	D9:0	$(1\ -2\ 5\ -4\ 2)$	0
D8:1	$(1\ -4\ 10\ -12\ 4)/2$	D9:1	$(1\ -6\ 17\ -24\ 14)$	1
* for degree 8, $a = (k + c)/4$		* for degree 9, $a = (k + c)/4$		

These relationships are summarised in the dimensional frame, Table 9.9.

Table 9.9: Dimension 4: Largest-known bipartite circulant graph families

Degree 8 families		Degree 9 families	
		D9:1	
	D8:0	↑	
D8:1		↓	
		D9:0	

A full set of LGMs and formulae for order and a generating set for graph families of both diameter classes is given in Appendix B.3.

For diameter $k = 2$, the graph of class 0 has order 14 and is not optimal. Instead there is one isomorphism class of extremal graph with order 18. For diameter $k = 3$, the graph of class 1 has order 62, whereas there are two isomorphism classes of extremal graph with order 70. For diameter $k = 4$, the graph of class 1 has order 194 compared with a single isomorphism class of extremal graph with order 198. Properties of the individual graphs up to diameter 16 are given in Appendix E, Tables E.7 and E.8.

9.6 Dimension 5, degrees 10 and 11

As the dimension is odd, there is translation between graph families of degree 10 and degree 11, and the graph families are defined for diameter classes modulo $f = 5$, so there are five diameter classes of largest-known bipartite circulant graph families for each degree.

For degree 10, each diameter class has one self-transpose graph family. The one for class 0, D10:0, is self-conjugate. The others form two conjugate pairs. For degree 11, each diameter class also has one self-transpose graph family. D11:3 is self-conjugate and is the translate of D10:0. Two others form a conjugate pair and are translates of one of the degree 10 conjugate pairs. The other two are neither conjugates nor translates.

The order polynomials are presented in Table 9.10 and the relationships are summarised in the dimensional frame, Table 9.11.

Table 9.10: Order of largest-known bipartite circulant graph families of degrees 10 and 11

Degree 10 families		Degree 11 families		c^*
Family	Order polynomial in $2a$	Family	Order polynomial in $2a$	
D10:0	$(1\ 0\ 6\ 0\ 5\ 0)/2$	D11:3	$(1\ 0\ 6\ 0\ 5\ 0)$	0
D10:1	$(1\ 4\ 13\ 20\ 14\ 4)/2$	D11:4	$(1\ 4\ 12\ 20\ 15\ 4)$	-2
D10:2	$(1\ -2\ 8\ -8\ 5\ -2)/2$	D11:0	$(1\ -2\ 8\ -8\ 5\ -2)$	1
D10:3	$(1\ 2\ 8\ 8\ 5\ 2)/2$	D11:1	$(1\ 2\ 8\ 8\ 5\ 2)$	-1
D10:4	$(1\ -4\ 13\ -20\ 14\ -4)/2$	D11:2	$(1\ -4\ 12\ -16\ 9\ -4)$	2

* for degree 10, $a = (2k + c)/5$ * for degree 11, $a = (2k + c - 1)/5$

Table 9.11: Dimension 5: Largest-known bipartite circulant graph families

Degree 10 families		Degree 11 families	
D10:3	\longleftrightarrow	D11:1	
\uparrow	D10:4	D11:2	\uparrow
\downarrow	\updownarrow	D11:3	\downarrow
	D10:0 \longleftrightarrow D11:0	D11:4	
D10:2	\longleftrightarrow	D11:0	

LGMs and formulae for order and a generating set for graph families of all diameter classes are given in Appendix B.4. For degree 10 and diameter 4, the graph in family D10:4 has order 282, whereas there is a larger graph, E10:4 with order 288. And for diameter 5, there is another graph, E10:5, with the same order 714 as the graph in family D10:0. For degree 11, for diameter 2 there is a graph E11:2 larger than the family graph. This is also true for diameter 3 and 4 with larger graphs E11:3 and E11:4. Properties of the individual graphs up to diameter 16 are given in Appendix E, Tables E.9 and E.10.

9.7 Conjectured order of extremal bipartite circulant graphs

In this chapter, the extremal and largest-known families of bipartite circulant graphs of degree 2 to 11 for arbitrary diameter k have been identified. We have seen that bipartite circulant graphs are related in various ways to their corresponding general circulant graphs. In particular, their order is expressed by a polynomial in k of the same degree, being equal to the dimension of the graph in each case. The first two coefficients of these polynomials is presented in Table 9.12.

From a comparison of the formulae for the order of the families of extremal and largest-known bipartite circulant graphs of degree 2 to 11, relationships may be discerned between the first and second terms and between the degrees, similar to the

Table 9.12: The first two coefficients in the polynomial formulae for the order of extremal and largest-known bipartite circulant graphs of degrees 2 to 11 and diameter k , for $k \geq k_d$ for some k_d

Dim f	Even degree $d = 2f$		Odd degree $d = 2f + 1$	
	Coeff of $(4k/f)^f$	Coeff of $(4k/f)^{f-1}$	Coeff of $(4k/f)^f$	Coeff of $(4k/f)^{f-1}$
1	1/2	0	1	-2
2	1/2	0	1	-2
3	1/2	0	1	-2
4	1/2	0	1	-2
5	1/2	0	1	-2

ones between general circulant graph families. Extrapolating these relationships to all higher degrees leads to the Extremal Order Conjecture for Bipartite Circulant Graphs, Conjecture 9.6, summarised in Table 9.13.

Conjecture 9.6. *Extremal Order Conjecture for Bipartite Circulant Graphs. Given an extremal bipartite circulant graph family of degree $d \geq 2$ and arbitrary diameter k , then the order of each graph in the family is given by a polynomial in k of degree f , the dimension of the graph. The leading term in the polynomial is $(1/2)(4/f)^f k^f$ for even degree, and $(4/f)^f k^f$ for odd degree. For the second term, the coefficient of k^{f-1} is zero for even degree and $(-2)(4/f)^{f-1}$ for odd. Graphs in such a family are extremal for all diameters $k \geq k_d$ for some threshold value k_d dependent on d .*

Table 9.13: The first two coefficients in the polynomial formulae for the conjectured order of extremal bipartite circulant graphs of degree d and diameter k , for $k \geq k_d$ for some k_d

Degree d	Dimension f	Coefficient of k^f	Coefficient of k^{f-1}
d even	$f = d/2$	$\frac{1}{2} \left(\frac{4}{f}\right)^f$	0
d odd	$f = (d-1)/2$	$\left(\frac{4}{f}\right)^f$	$-2 \left(\frac{4}{f}\right)^{f-1}$

The conjecture is true for dimensions 1 and 2, with $k_d = 1$. For dimension 3, it is true for the families of largest-known circulant graphs with $k_d = 1$, and for dimensions 4 and 5 with $k_d = 5$.

Comparing the conjectured order of extremal bipartite circulant graph families with the general circulant Extremal Order Conjecture, 3.1, the leading coefficients for both even and odd degree are identical. The difference in the second coefficient is $(4/f)^{f-1}$ for even degree and $2(4/f)^{f-1}$ for odd. This relationship is a consequence of Theorems 9.1 to Theorem 9.3 and the fact that corresponding general and bipartite circulant graph families share common LGMS.

CHAPTER 10

ABELIAN CAYLEY GRAPH FAMILIES OF HIGHER CYCLIC RANK

Chapter 10 presents largest-known non-circulant Abelian Cayley graph families up to degree 15, and one of degree 19, all quasimaximal.

Circulant graphs are simply a special case of Abelian Cayley graphs, where the cyclic rank is 1. The cyclic rank of an Abelian Cayley graph is at most equal to the dimension of the generating set. Equality is achieved only when each generator defines a distinct cyclic subgroup of the associated Abelian group. Depending on the order of the group, the number of potential generating sets for non-circulant Abelian Cayley graphs (of cyclic rank greater than 1) may be some orders of magnitude more than for circulant graphs of the same order. Therefore, a simple search for extremal Abelian Cayley graphs of given degree and diameter becomes intractable computationally even earlier than for circulant graphs.

The approach employed for circulant graph families of degree 12 and above, using canonical quasimaximal LGMs (LGM odd bases, for even degree), is based on a bijection between lattices in \mathbb{Z}^f and Abelian Cayley graphs, and so is not limited to circulant graphs in particular. Therefore, it is reasonable to consider this approach in the search for Abelian Cayley graph families of cyclic rank 2 or more.

As described in Chapter 9, this search method proceeds by defining a candidate LGM, taking its adjoint, and then post-multiplying by a sequence of simple column vectors in turn. Each resulting candidate generating set is then tested for validity before moving to the next. The enhancement required for Abelian Cayley graph families of higher cyclic rank is to replace the column vector by a matrix with the number of columns equal to the cyclic rank. In this way, the candidate generators have the correct cyclic rank, so that the vector representing each generator has number of elements equal to the cyclic rank. The complexity of the search algorithm increases rapidly with cyclic rank, in step with the number of elements in the post-multiplying matrix (being the product of dimension and cyclic rank). Therefore, it was only possible to explore Abelian Cayley graph families up to degree 15 with the available computing resource (compared with degree 20 for circulant graph families). The

searches were extensive but not exhaustive, so it is possible that these largest-known Abelian Cayley graph families are not extremal. Also, the procedure only searched for quasimaximal graph families, and for each degree there is a threshold diameter below which the extremal graphs do not belong to quasimaximal families. (This is discussed further in Chapter 13.) Therefore, even in case a graph family is extremal, its members below the threshold diameter will not be extremal graphs.

Each largest-known non-circulant Abelian Cayley graph family described in this chapter is comprehensively documented in Appendix C, including its LGM and formulae for its order and a generating set. The existence of these families for all diameters in their respective diameter classes is confirmed by the Abelian Cayley graph families Existence Proof Theorem 6.16. In addition, each graph up to diameter 16 is documented in Appendix F.

Despite the fact that these families exist, it has not always been possible to identify generating set formulae that are valid for every diameter in the class. It often occurs that the formulae for a set of generators all share a common factor with the order formula for every diameter within a subclass of the diameter class. For those diameters, the formulae will fail to generate a graph. In this chapter and in Appendix C, the aim has been to find sets of formulae that avoid such a common factor. Where this has not been possible, then a set with the largest factor found has been selected. In Appendix C, this is identified at the head of each table. In some cases it has not proved possible to discover any set of formulae for the generating sets of a family, such as for degree 13, diameter class $0 \pmod{3}$. More extensive computer searches may well fill these gaps in future.

It is immediate that the non-circulant Abelian Cayley graphs cannot exist below degree 4, as they must have dimension at least 2 in order to have cyclic rank above 1. Therefore, the extremal circulant graph families of degrees 2 and 3 are also extremal Abelian Cayley graph families.

10.1 Dimension 2, degrees 4 and 5

As the dimension is even, there is no translation between degrees 4 and 5, and the graph families are defined for diameter classes modulo $h = f/2 = 1$, giving just one class per degree.

For degree 4, the largest-known Abelian Cayley graph family is circulant. Degree 5 is the first degree where non-circulant Abelian Cayley graphs are extremal. The extremal cyclic-rank 2 graph family has the same order as the extremal circulant

graph family. Their orders are shown in Table 10.1. See Appendices C.1 and F Table F.1 for details.

Table 10.1: Order and cyclic orders of extremal Abelian Cayley graph families of degrees 4 and 5

Degree (all diameters)	Order polynomials in $2a$	
	Circulant	Non-circulant (with cyclic orders)
Degree 4 $a = k$	$(1\ 2\ 2)/2$	$(1\ 0\ 0)/2 = (1\ 0) \times (1\ 0)/2$
Degree 5 $a = k$	$(1\ 0\ 0)$	$(1\ 0\ 0) = (1\ 0) \times (1\ 0)$

10.2 Dimension 3, degrees 6 and 7

As the dimension is odd, there is translation between degrees 6 and 7, and the graph families are defined for diameter classes modulo $f = 3$, giving three classes per degree.

For degrees 6 and 7, the largest-known Abelian Cayley graph families are all circulant. The largest-known non-circulant Abelian Cayley graph families have cyclic rank 2 or 3, depending on the diameter class. For degree 6 diameter class 1 and degree 7 diameter class 0, there are two non-isomorphic largest-known non-circulant Abelian Cayley graph families: one with cyclic rank 2 and one with cyclic rank 3 (translate pairs). Their orders are shown in Tables 10.2 and 10.3.

Table 10.2: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 6

Diameter class $k \pmod{3}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = 2k/3$	$(1\ 2\ 3\ 2)/2$	$(1\ 2\ 0\ 0)/2$ $= (1\ 2\ 0)/2 \times (1\ 0)$
$k \equiv 1$ $a = (2k + 1)/3$	$(1\ 0\ 3\ 0)/2$	$(1\ 0\ 0\ 0)/2$ $= (1\ 0\ 0) \times (1\ 0)/2$ $(1\ 0\ 0\ 0)/2$ $= (1\ 0) \times (1\ 0) \times (1\ 0)/2$
$k \equiv 2$ $a = (2k - 1)/3$	$(1\ -2\ 3\ -2)/2$	$(1\ -2\ 0\ 0)/2$ $= (1\ -2\ 0)/2 \times (1\ 0)$

Table 10.3: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 7

Diameter class $k \pmod{3}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = 2k/3$	$(1\ 0\ 3\ 0)$	$(1\ 0\ 0\ 0)$ $= (1\ 0\ 0) \times (1\ 0)$ $(1\ 0\ 0\ 0)$ $= (1\ 0) \times (1\ 0) \times (1\ 0)$
$k \equiv 1$ $a = (2k+1)/3$	$(1\ -2\ 3\ -2)$	$(1\ -2\ 2\ 0)$ $= (1\ -2\ 2\ 0)/2 \times (2)$
$k \equiv 2$ $a = (2k-1)/3$	$(1\ 2\ 3\ 2)$	$(1\ 2\ 2\ 0)$ $= (1\ 2\ 2\ 0)/2 \times (2)$

10.3 Dimension 4, degrees 8 and 9

As the dimension is even, there is no translation between degrees 8 and 9, and the graph families are defined for diameter classes modulo $h = f/2 = 2$, giving two classes per degree.

For degree 8, the largest-known Abelian Cayley graph families are both circulant. However, as an exception, the extremal Abelian Cayley graph of degree 8 and diameter 2 has cyclic rank 2. It has order 36, with cyclic orders 12 and 3, compared with 35 for the extremal circulant graph (see Appendix F Table F.2 and Appendix D Table D.7). The largest-known non-circulant Abelian Cayley graph families have cyclic rank 2. Their orders are shown in Table 10.4.

Table 10.4: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 8

Diameter class $k \pmod{2}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = k/2$	$(1\ 2\ 6\ 4\ 0)/2$	$(1\ 2\ 3\ 4\ 2)/2$ $= (1\ 1\ 2\ 2)/2 \times (1\ 1)$
$k \equiv 1$ $a = (k+1)/2$	$(1\ -2\ 6\ -4\ 0)/2$	$(1\ -2\ 3\ -4\ 2)/2$ $= (1\ -1\ 2\ -2)/2 \times (1\ -1)$

In contrast, for degree 9, the largest-known Abelian Cayley graph families are both non-circulant. This is the lowest degree where a non-circulant Abelian Cayley graph family has larger order polynomial than the largest-known circulant graph family. Their orders are shown in Table 10.5. See Appendices C.2 and F Table F.3 for details.

Table 10.5: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 9

Diameter class $k \pmod{2}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = k/2$	$(1\ 0\ 3\ 2\ 0)$	$(1\ 0\ 4\ 0\ 0)$ $= (1\ 0\ 4\ 0)/2 \times (2\ 0)$
$k \equiv 1$ $a = k/2$	$(1\ 0\ 3\ 0\ 0)$	$(1\ 0\ 4\ 0\ -1)$ $= (1\ 0\ 4\ 0\ -1)/2 \times (2)$

10.4 Dimension 5, degrees 10 and 11

As the dimension is odd, there is translation between degrees 10 and 11, and the graph families are defined for diameter classes modulo $f = 5$, giving five classes per degree.

Of the five largest-known Abelian Cayley graph families of degree 10, two are circulant, two have cyclic rank 2 and one has cyclic rank 3. The largest-known families of degree 11 are the translates of these families for each class. Their orders are shown in Tables 10.6 and 10.7. See Appendices C.3 and F Table F.4 and F.5 for details.

Table 10.6: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 10

Diameter class $k \pmod{5}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = 2k/5$	$(1\ 2\ 8\ 8\ 5\ 2)/2$	$(1\ 2\ 7\ 10\ 0\ 0)/2$ $= (1\ 2\ 7\ 10\ 0)/2 \times (1\ 0)$
$k \equiv 1$ $a = (2k + 3)/5$	$(1\ -4\ 13\ -20\ 14\ -4)/2$	$(1\ -4\ 14\ -24\ 17\ -4)/2$ $= (1\ -3\ 11\ -13\ 4)/2 \times (1\ -1)$
$k \equiv 2$ $a = (2k + 1)/5$	$(1\ 0\ 6\ 0\ 5\ 0)/2$	$(1\ 0\ 7\ 0\ 0\ 0)/2$ $= (1\ 0\ 7\ 0) \times (1\ 0) \times (1\ 0)/2$
$k \equiv 3$ $a = (2k - 1)/5$	$(1\ 4\ 13\ 20\ 14\ 4)/2$	$(1\ 4\ 14\ 24\ 17\ 4)/2$ $= (1\ 3\ 11\ 13\ 4)/2 \times (1\ 1)$
$k \equiv 4$ $a = (2k + 2)/5$	$(1\ -2\ 8\ -8\ 5\ -2)/2$	$(1\ -2\ 7\ -6\ 0\ 0)/2$ $= (1\ -2\ 7\ -6\ 0)/2 \times (1\ 0)$

Table 10.7: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 11

Diameter class $k \pmod{5}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = 2k/5$	(1 0 6 0 5 0)	(1 0 7 0 0 0) $= (1 0 7 0) \times (1 0) \times (1 0)$
$k \equiv 1$ $a = (2k - 2)/5$	(1 4 13 20 14 4)	(1 4 14 24 17 4) $= (1 3 11 13 4) \times (1 1)$
$k \equiv 2$ $a = (2k + 1)/5$	(1 -2 8 -8 5 -2)	(1 -2 7 -6 2 0) $= (1 -2 7 -6 2 0)/2 \times (2)$
$k \equiv 3$ $a = (2k - 1)/5$	(1 2 8 8 5 2)	(1 2 7 10 0 0) $= (1 2 7 10 0)/2 \times (1 0) \times (2)$
$k \equiv 4$ $a = (2k + 2)/5$	(1 -4 12 -16 9 -4)	(1 -4 14 -24 17 -4) $= (1 -3 11 -13 4) \times (1 -1)$

10.5 Dimension 6, degrees 12 and 13

As the dimension is even, there is no translation between degrees 12 and 13, and the graph families are defined for diameter classes modulo $h = f/2 = 3$, giving three classes per degree.

For degree 12, the largest known Abelian Cayley graph families are all non-circulant, with cyclic rank 2. See Table 10.8 and Appendix F Table F.6 for details.

Table 10.8: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 12

Diameter class $k \pmod{3}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = k/3$	(1 2 11 14 13 6 0)/2	(1 2 12 16 10 0 0)/2 $= (1 2 12 16 10 0)/2 \times (1 0)$
$k \equiv 1$ $a = (k - 1)/3$	(1 6 24 58 75 46 10)/2	(1 6 25 60 72 40 8)/2 $= (1 5 20 40 32 8)/2 \times (1 1)$
$k \equiv 2$ $a = (k + 1)/3$	(1 -2 11 -12 -2 4 0)/2	(1 -2 12 -16 10 0 0)/2 $= (1 -2 12 -16 10 0)/2 \times (1 0)$

For degree 13, they are also non-circulant. See Table 10.9 and Appendix F Table F.7 for details. For diameters $k \equiv 1$ and 2, the families are a conjugate pair with cyclic rank 2. For diameter $k \equiv 0$, formulae for two different orders are included in the table: (1 0 10 0 0 0 0) and (1 0 12 0 0 0 0). The first is the largest-known family for which formulae have also been discovered for generating sets. For the second, it has not yet been possible to identify generating sets that can be represented by such

formulae, although graphs with this order have been discovered for diameters up to 21 and included in Appendix F. The order $(1\ 0\ 12\ 0\ 0\ 0\ 0)$ is associated with a graph family with cyclic rank 4. However, the common factors shared by the cyclic orders leads to a division into two subfamilies with distinct cyclic order formulae:

$(1\ 0\ 12\ 0)/2 \times (2\ 0) \times (1\ 0) \times (1\ 0)$ for $k \equiv 0 \pmod{6}$, and $(1\ 0\ 12\ 0)/4 \times (4\ 0) \times (1\ 0) \times (1\ 0)$ for $k \equiv 3 \pmod{6}$. Their common LGM is presented in Appendix C.4 which contains details of all the Abelian Cayley graph families of degrees 12 and 13.

Table 10.9: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 13

Diameter class $k \pmod{3}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = k/3$	$(1\ 0\ 8\ 2\ -1\ -4\ 0)/2$	$(1\ 0\ 10\ 0\ 0\ 0\ 0)$ $= (1\ 0\ 10\ 0\ 0) \times (1\ 0) \times (1\ 0)$
$k \equiv 0 \pmod{6}$		$(1\ 0\ 12\ 0\ 0\ 0\ 0)$ $= (1\ 0\ 12\ 0)/2 \times (2\ 0) \times (1\ 0) \times (1\ 0)$
$k \equiv 3 \pmod{6}$		$(1\ 0\ 12\ 0\ 0\ 0\ 0)$ $= (1\ 0\ 12\ 0)/4 \times (4\ 0) \times (1\ 0) \times (1\ 0)$
$k \equiv 1$ $a = (k-1)/3$	$(1\ 4\ 16\ 30\ 29\ 16\ 4)$	$(1\ 4\ 17\ 34\ 28\ 8\ 0)$ $= (1\ 3\ 14\ 20\ 8\ 0)/4 \times (4\ 4)$
$k \equiv 2$ $a = (k+1)/3$	$(1\ -4\ 16\ -30\ 29\ -16\ 4)$	$(1\ -4\ 17\ -34\ 28\ -8\ 0)$ $= (1\ -3\ 14\ -20\ 8\ 0)/4 \times (4\ -4)$

10.6 Dimension 7, degrees 14 and 15

As the dimension is odd, there is translation between degrees 14 and 15, and the graph families are defined for diameter classes modulo $f = 7$, giving seven classes per degree.

Of the seven largest-known Abelian Cayley graph families of degree 14, just one is circulant and the rest have cyclic rank 2. For degree 15, none of the largest-known families are circulant: six have cyclic rank 3 and the other has cyclic rank 5. Their orders are shown in Tables 10.10 and 10.11. Details of the graph families and the graphs up to diameter 16 are given in Appendices C.5 and F Tables F.8 and F.9 for details.

Table 10.10: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 14

Diameter class $k \pmod{7}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = 2k/7$	$(1\ 2\ 15\ 20\ 21\ 12\ 4\ 0)/2$	$(1\ 2\ 16\ 20\ 13\ 6\ 0\ 0)/2$ $= (1\ 2\ 16\ 20\ 13\ 6\ 0)/2 \times (1\ 0)$
$k \equiv 1$ $a = (2k - 2)/7$	$(1\ 6\ 28\ 76\ 127\ 126\ 67\ 14)/2$	$(1\ 6\ 30\ 84\ 113\ 70\ 16\ 0)/2$ $= (1\ 5\ 25\ 59\ 54\ 16\ 0)/8 \times (4\ 4)$
$k \equiv 2$ $a = (2k + 3)/7$	$(1\ -4\ 21\ -46\ 50\ -30\ 8\ 0)/2$	$(1\ -4\ 22\ -48\ 41\ -12\ 0\ 0)/2$ $= (1\ -3\ 19\ -29\ 12\ 0)/2 \times (1\ -1\ 0)$
$k \equiv 3$ $a = (2k + 1)/7$	$(1\ 0\ 14\ 0\ 21\ 0\ 7\ 0)/2$	$(1\ 0\ 13\ -2\ 7\ 2\ 0\ 0)/2$ $= (1\ 0\ 13\ -2\ 7\ 2\ 0)/2 \times (1\ 0)$
$k \equiv 4$ $a = (2k - 1)/7$	$(1\ 4\ 21\ 46\ 50\ 30\ 8\ 0)/2$	$(1\ 4\ 22\ 48\ 41\ 12\ 0\ 0)/2$ $= (1\ 3\ 19\ 29\ 12\ 0)/2 \times (1\ 1\ 0)$
$k \equiv 5$ $a = (2k + 4)/7$	$(1\ -6\ 28\ -76\ 127\ -126\ 67\ -14)/2$	$(1\ -6\ 30\ -84\ 113\ -70\ 16\ 0)/2$ $= (1\ -5\ 25\ -59\ 54\ -16\ 0)/8 \times (4\ -4)$
$k \equiv 6$ $a = (2k + 2)/7$	$(1\ -2\ 15\ -20\ 21\ -12\ 4\ 0)/2$	$(1\ -2\ 16\ -20\ 13\ -6\ 0\ 0)/2$ $= (1\ -2\ 16\ -20\ 13\ -6\ 0)/2 \times (1\ 0)$

Table 10.11: Order and cyclic orders of largest-known Abelian Cayley graph families of degree 15

Diameter class $k \pmod{7}$	Order polynomials in $2a$ for diameter class k	
	Circulant	Non-circulant (with cyclic orders)
$k \equiv 0$ $a = 2k/7$	$(1\ 0\ 14\ 0\ 21\ 0\ 7\ 0)$	$(1\ 0\ 15\ 0\ 0\ 0\ 0\ 0)$ $= (1\ 0\ 15\ 0) \times (1\ 0) \times (1\ 0) \times (1\ 0) \times (1\ 0)$
$k \equiv 1$ $a = (2k - 2)/7$	$(1\ 4\ 20\ 44\ 57\ 44\ 19\ 4)$	$(1\ 4\ 22\ 48\ 41\ 12\ 0\ 0)$ $= (1\ 3\ 19\ 29\ 12\ 0\ 0)/16 \times (4\ 4) \times (4)$
$k \equiv 2$ $a = (2k + 3)/7$	$(1\ -6\ 28\ -76\ 127\ -126\ 67\ -14)$	$(1\ -6\ 30\ -84\ 113\ -70\ 16\ 0)$ $= (1\ -5\ 25\ -59\ 54\ -16\ 0)/4 \times (2\ -2) \times (2)$
$k \equiv 3$ $a = (2k + 1)/7$	$(1\ -2\ 14\ -16\ 11\ -6\ 3\ -2)$	$(1\ -2\ 16\ -20\ 13\ -6\ 0\ 0)$ $= (1\ -2\ 16\ -20\ 13\ -6\ 0)/2 \times (1\ 0) \times (2)$
$k \equiv 4$ $a = (2k - 1)/7$	$(1\ 2\ 14\ 20\ 27\ 18\ 11\ 2)$	$(1\ 2\ 16\ 20\ 13\ 6\ 0\ 0)$ $= (1\ 2\ 16\ 20\ 13\ 6\ 0)/2 \times (1\ 0) \times (2)$
$k \equiv 5$ $a = (2k - 3)/7$	$(1\ 6\ 28\ 76\ 127\ 126\ 67\ 14)$	$(1\ 6\ 30\ 84\ 113\ 70\ 16\ 0)$ $= (1\ 5\ 25\ 59\ 54\ 16\ 0)/4 \times (2\ 2) \times (2)$
$k \equiv 6$ $a = (2k + 2)/7$	$(1\ -4\ 20\ -44\ 57\ -44\ 19\ -4)$	$(1\ -4\ 22\ -48\ 41\ -12\ 0\ 0)$ $= (1\ -3\ 19\ -29\ 12\ 0\ 0)/16 \times (4\ -4) \times (4)$

10.7 Dimension 9, degrees 19

The systematic search for extremal Abelian Cayley graph families only reached degree 15. However, the investigation of a dimension 9 LGM with a format that could be extrapolated to higher dimensions resulted in the discovery of a quasimaximal Abelian Cayley graph family with order larger than the largest-known circulant graph family. It has diameter class $k \equiv 0 \pmod{9}$. For diameters $k \equiv 9$ and $18 \pmod{27}$, it has cyclic rank 7. LGM, order and generating set polynomials are presented in Appendix C, Table C.30. For diameters $k \equiv 0 \pmod{27}$, the values of these polynomials have common divisor 9 and therefore do not generate the graph. For this case, the cyclic rank and generating set have not yet been discovered. The order polynomial is shown in Table 10.12 alongside the corresponding largest-known circulant graph family.

Table 10.12: Order and cyclic orders of largest-known Abelian Cayley graph family of degree 19, diameter class 0

Order polynomials in $2a$, $a = 2k/9$	
Circulant	Non-circulant (with cyclic orders*)
$(1\ 0\ 20\ 0\ 58\ 0\ 43\ 0\ 9\ 0)$	$(1\ 0\ 27\ 0\ 0\ 0\ 0\ 0\ 0)$
	$= (1\ 0\ 27\ 0) \times (1\ 0) \times (1\ 0) \times (1\ 0) \times (1\ 0) \times (1\ 0) \times (1\ 0)$

* cyclic orders only for diameter $k \equiv 9$ and $18 \pmod{27}$

ARC-TRANSITIVE CIRCULANT GRAPH FAMILIES FROM LUCAS POLYNOMIALS

Chapter 11 illustrates a surprising and beautiful relationship between Lucas polynomials and an infinite sequence of quasimaximal circulant graph families that are arc-transitive and have multiplicative generating sets. For any dimension, the order and generating sets of these families are defined in terms of Lucas polynomials.

Lucas polynomials (OEIS:A162514, [42]) are a generalisation of the Fibonacci sequence developed by Lucas [31]. They are defined recursively: $L_0(x) = 2$, $L_1(x) = x$, $L_f(x) = xL_{f-1}(x) + L_{f-2}(x)$ for $f > 1$. They are also sometimes called circulant Lucas polynomials. When $x = 1$, they reduce to give the Lucas number sequence. An alternative combinatorial definition of Lucas numbers is that $L_f(1)$ is the number of matchings in a cycle on f vertices.

The first ten Lucas polynomials are shown in Table 11.1 along with their coefficient representation in vector format. For simplicity of presentation, we adopt the shorthand notation L_f for the coefficient representation of $L_f(x)$, as shown in the final column of the table.

Table 11.1: The first ten Lucas polynomials and their coefficient representation

f	Lucas polynomials $L_f(x)$	Coefficient representation L_f
0	2	(2)
1	x	(1 0)
2	$x^2 + 2$	(1 0 2)
3	$x^3 + 3x$	(1 0 3 0)
4	$x^4 + 4x^2 + 2$	(1 0 4 0 2)
5	$x^5 + 5x^3 + 5x$	(1 0 5 0 5 0)
6	$x^6 + 6x^4 + 9x^2 + 2$	(1 0 6 0 9 0 2)
7	$x^7 + 7x^5 + 14x^3 + 7x$	(1 0 7 0 14 0 7 0)
8	$x^8 + 8x^6 + 20x^4 + 16x^2 + 2$	(1 0 8 0 20 0 16 0 2)
9	$x^9 + 9x^7 + 27x^5 + 30x^3 + 9x$	(1 0 9 0 27 0 30 0 9 0)

For $f > 0$, the first two terms have coefficient 1 and 0. This is consistent with the order polynomials in $2a$ for f -dimensional quasimaximal circulant graph families of the principal diameter class for each degree. For odd degree $d = 2f + 1$, diameter

class $k \equiv 0 \pmod{f}$, $a = 2k/f$; and in case of odd dimension, for even degree $d = 2f$, diameter class $k \equiv (f-1)/2 \pmod{f}$, $a = (2k+1)/f$. For odd dimension f , we will present constructions for circulant graph families with order polynomials given by both these sets of parameters. In fact, it emerges that for each such family of dimension f , the formulae for all the generators are constructed from the Lucas polynomials of all lower indices. These graphs are denoted *Lucas circulant graphs*. Note that for $f > 1$, the third coefficient is f .

For the even-degree Lucas family of order $n = L_f/2$ and diameter class $k \equiv (f-1)/2 \pmod{f}$, the generators g_i are defined in terms of L_i as follows:

$$\begin{aligned} g_1 &= L_0/2 \\ g_{2i} &= (L_{(f-1)/2+i} - L_{(f+1)/2-i})/2 \\ g_{2i+1} &= (L_{(f-1)/2+i} + L_{(f+1)/2-i})/2 \end{aligned}$$

for $1 \leq i \leq (f-1)/2$, where the L_i are polynomials in $2a$ for $a = (2k+1)/f$.

An example for dimension $f = 9$, diameter class $k \equiv 4 \pmod{9}$ is given in Table 11.2. Here, the order $n = L_9/2$, and the generators are all sums or differences of $L_0/2$ to $L_8/2$, taken as polynomials in $2a$, with $a = (2k+1)/9$.

Table 11.2: Example: Lucas family of dimension $f = 9$, degree $d = 18$

Order/generator	L_i	Polynomial in $2a$ for $a = (2k+1)/9$
Order, n	$L_9/2$	$(1\ 0\ 9\ 0\ 27\ 0\ 30\ 0\ 9\ 0)/2$
g_1	$L_0/2$	$(2)/2$
g_8 & g_9	$L_8/2$	$(1\ 0\ 8\ 0\ 20\ 0\ 16\ 0\ 2)/2$
	$\mp L_1/2$	$\mp (1\ 0)/2$
g_6 & g_7	$L_7/2$	$(1\ 0\ 7\ 0\ 14\ 0\ 7\ 0)/2$
	$\mp L_2/2$	$\mp (1\ 0\ 2)/2$
g_4 & g_5	$L_6/2$	$(1\ 0\ 6\ 0\ 9\ 0\ 2)/2$
	$\mp L_3/2$	$\mp (1\ 0\ 3\ 0)/2$
g_2 & g_3	$L_5/2$	$(1\ 0\ 5\ 0\ 5\ 0)/2$
	$\mp L_4/2$	$\mp (1\ 0\ 4\ 0\ 2)/2$

For the odd-degree Lucas family of order $n = L_f$, the first generator remains $g_1 = L_0/2 = 1$. The formulae for the subsequent $(f-1)/2$ pairs of generators are just the complements in $L_f/2$ of the even-degree generator pairs. Thus,
 $g_{2i} = (L_f - L_{(f-1)/2+i} + L_{(f+1)/2-i})/2$ and $g_{2i+1} = (L_f - L_{(f-1)/2+i} - L_{(f+1)/2-i})/2$,
for $i = 1, \dots, (f-1)/2$. These are all polynomials in $2a$, with $a = 2k/f$.

All these Lucas graph families are quasimaximal, as determined by the first two coefficients of their order polynomials, and have maximal odd girth, $2k+1$, where k is

the diameter. For degrees $d > 20$, they are new largest-known circulant graph families, thereby improving the lower bound for these degrees and diameter classes.

For the even-degree Lucas families, we will now show that their generating sets are multiplicative and that the graphs are therefore arc-transitive. First, we state and prove a lemma from the literature, for example see [12].

Lemma 11.1. *For $m \geq n$, $L_{m+n} = L_m L_n - (-1)^n L_{m-n}$, where L_i is the i th Lucas polynomial.*

Proof. For $n = 0$, noting that $L_0 = 2$, we have

$L_m L_n - (-1)^n L_{m-n} = 2L_m - L_m = L_m$, proving the lemma for $n = 0$. For $n = 1$, we note that $L_1 = x$. Then $L_m L_n - (-1)^n L_{m-n} = xL_m + L_{m-1} = L_{m+1}$ by definition, proving the lemma for $n = 1$. Suppose the lemma is true for any $n \leq N$. Then

$$\begin{aligned}
 L_{m+(N+1)} &= L_{(m+1)+N} \\
 &= L_{m+1} L_N - (-1)^N L_{m+1-N} \\
 &= (xL_m + L_{m-1}) L_N - (-1)^N L_{m+1-N} \\
 &= L_m(L_{N+1} - L_{N-1}) + L_{m-1} L_N - (-1)^N L_{m+1-N} \\
 &= L_m L_{N+1} + L_{m-1} L_N - L_m L_{N-1} - (-1)^N L_{m+1-N} \\
 &= L_m L_{N+1} + L_{m+N-1} + (-1)^N L_{m-(N+1)} - L_{m+N-1} \\
 &\quad - (-1)^{N-1} L_{m-(N-1)} - (-1)^N L_{m-(N-1)} \\
 &= L_m L_{N+1} - (-1)^{N+1} L_{m-(N+1)}
 \end{aligned}$$

proving the lemma for $n = N + 1$. □

Theorem 11.2. *The generators for the Lucas circulant graph families of dimension $f \equiv 1 \pmod{4}$ may all be expressed as powers of the second generator $g_2 = (L_{(f+1)/2} - L_{(f-1)/2})/2$, as shown in Table 11.3.*

Table 11.3: Generators for the Lucas family of dimension $f = 4m + 1$ for any $m \geq 1$

Subscript class	Generator $i = 1, \dots, m$	In terms of Lucas polynomials	As a power of g_2
2	g_{4i-2}	$(L_{(f+4i-3)/2} - L_{(f-4i+3)/2})/2$	g_2^{4i-3}
3	g_{4i-1}	$(L_{(f+4i-3)/2} + L_{(f-4i+3)/2})/2$	g_2^{f-4i+3}
0	g_{4i}	$(L_{(f+4i-1)/2} - L_{(f-4i+1)/2})/2$	g_2^{f-4i+1}
1	g_{4i+1}	$(L_{(f+4i-1)/2} + L_{(f-4i+1)/2})/2$	g_2^{4i-1}

Proof. The proof proceeds by induction on i for each of the four subscript classes of generators in Table 11.3, making extensive use of Lemma 11.1. All the calculations are modulo n .

For class 2, $g_2 = (L_{(f+1)/2} - L_{(f-1)/2})/2$, by definition, is the initial generator.

For the initial generator for class 0, we need to show that $g_2^2 = \pm g_{f-1} \pmod{n}$. We have

$$\begin{aligned} g_2^2 &= (L_{(f+1)/2} - L_{(f-1)/2})^2/4 \\ &= (L_{(f+1)/2}L_{(f+1)/2} - 2L_{(f+1)/2}L_{(f-1)/2} + L_{(f-1)/2}L_{(f-1)/2})/4 \\ &= (L_{f+1} - 2 - 2(L_f + L_1) + L_{f-1} + 2)/4 \\ &= (L_f L_1 + L_{f-1} - 2L_f - 2L_1 + L_{f-1})/4. \end{aligned}$$

Now $L_f/2 = n$ and L_1 is even, so that $(L_1 - 2)L_f/4 \equiv 0 \pmod{n}$. Hence

$$g_2^2 = (L_{f-1} - L_1)/2 = g_{f-1}.$$

For the initial generator for class 1, we must show that $g_2^3 = \pm g_5 \pmod{n}$. Now

$$\begin{aligned} g_2^3 &= g_2 g_2^2 \\ &= (L_{(f+1)/2} - L_{(f-1)/2})(L_{f-1} - L_1)/4 \\ &= (L_{(f+1)/2}L_{f-1} - L_{(f+1)/2}L_1 - L_{(f-1)/2}L_{f-1} + L_{(f-1)/2}L_1)/4 \\ &= (L_{(3f-1)/2} - L_{(f-3)/2} - L_{(f+3)/2} + L_{(f-1)/2} - L_{(3f-3)/2} - L_{(f-1)/2} \\ &\quad + L_{(f+1)/2} - L_{(f-3)/2})/4 \\ &= (L_f L_{(f-1)/2} - L_{(f+1)/2} - L_{(f-3)/2} - L_{(f+3)/2} + L_{(f-1)/2} - L_f L_{(f-3)/2} \\ &\quad - L_{(f+3)/2} - L_{(f-1)/2} + L_{(f+1)/2} - L_{(f-3)/2})/4 \\ &= (L_f(L_{(f-1)/2} - L_{(f-3)/2}) - 2L_{(f+3)/2} - 2L_{(f-3)/2})/4. \end{aligned}$$

Again we have $(L_f(L_{(f-1)/2} - L_{(f-3)/2}))/4 \equiv 0 \pmod{n}$. Hence

$$g_2^3 = -(L_{(f+3)/2} + L_{(f-3)/2})/2 = -g_5.$$

For the initial generator for class 3, we must show that $g_2^4 = \pm g_{f-2} \pmod{n}$. So

$$\begin{aligned} g_2^4 &= g_2^2 g_2^2 = g_{f-1} g_{f-1} \\ &= (L_{f-1} - L_1)(L_{f-1} - L_1) \\ &= (L_{f-1}L_{f-1} - 2L_{f-1}L_1 + L_1L_1)/4 \\ &= (L_{2f-2} + 2 - 2(L_f - L_{f-2}) + L_2 - 2)/4 \\ &= (L_f L_{f-2} + L_2 + 2 - 2L_f + 2L_{f-2} + L_2 - 2)/4 \\ &= ((L_{f-2} - 2)L_f + 2L_{f-2} + 2L_2)/4. \end{aligned}$$

We have $(L_{f-2} - 2)L_f/4 \equiv 0 \pmod{n}$. Hence, $g_2^4 = (L_{f-2} + L_2)/2 = g_{f-2}$.

Having established the initial generator for all four subscript classes, we now need to consider the inductive step for each. The proof for class 2 is presented below. The other three proofs are similar. For class 2, we must show that $g_2^4 g_{4i-2} = g_{4(i+1)-2}$ for any i . We have

$$\begin{aligned}
g_2^4 g_{4i-2} &= g_{f-2} g_{4i-2} \\
&= (L_{f-2} + L_2)(L_{(f+4i-3)/2} - L_{(f-4i+3)/2})/4 \\
&= (L_{f-2} L_{(f+4i-3)/2} - L_{f-2} L_{(f-4i+3)/2} + L_2 L_{(f+4i-3)/2} \\
&\quad - L_2 L_{(f-4i+3)/2})/4 \\
&= (L_{(3f+4i-7)/2} - L_{(f-4i-1)/2} - L_{(3f-4i-1)/2} - L_{(f+4i-7)/2} \\
&\quad + (L_{(f+4i+1)/2} + L_{(f+4i-7)/2} - L_{(f-4i+7)/2} - L_{(f-4i-1)/2})/4 \\
&= (L_f L_{(f+4i-7)/2} + L_{(f-4i+7)/2} - L_{(f-4i-1)/2} - L_f L_{(f-4i-1)/2} \\
&\quad + L_{(f+4i+1)/2} - L_{(f+4i-7)/2} + (L_{(f+4i+1)/2} + L_{(f+4i-7)/2} \\
&\quad - L_{(f-4i+7)/2} - L_{(f-4i-1)/2})/4 \\
&= (L_f (L_{(f+4i-7)/2} - L_{(f-4i-1)/2}) + 2L_{(f+4i+1)/2} - 2L_{(f-4i-1)/2})/4
\end{aligned}$$

We have $L_f (L_{(f+4i-7)/2} - L_{(f-4i-1)/2}) \equiv 0 \pmod{n}$. Hence

$$g_2^4 g_{4i-2} = (L_{(f+4i+1)/2} - L_{(f-4i-1)/2})/2 = g_{4(i+1)-2}. \quad \square$$

We now establish that the Lucas circulant graph families considered in Theorem 11.2 have a multiplicative generating set and are therefore arc-transitive.

Theorem 11.3. *The even-degree Lucas circulant graph families of dimension $f \equiv 1 \pmod{4}$ have multiplicative generating set $\{1, g_2, g_2^2, \dots, g_2^{f-1}\}$.*

Proof. It has already been established that the generators are all powers of g_2 , using the notation of Theorem 11.2. It only remains to demonstrate that $|g_2^f| = 1$. Using the relationships of generators in class 2 in Table 11.3, $g_2^f = (L_f - L_0)/2 = -1$. \square

The Lucas circulant graph families of dimension $f \equiv 3 \pmod{4}$ are proved to have multiplicative generating sets and are therefore arc-transitive using the same approach as for dimension $f \equiv 1 \pmod{4}$. Only the statements of the theorems are given here, but their proofs follow the structure for the first case exactly.

Theorem 11.4. *The generators for the Lucas circulant graph families of dimension $f \equiv 3 \pmod{4}$ may all be expressed as powers of the second generator $g_2 = (L_{(f+1)/2} - L_{(f-1)/2})/2$, as shown in Table 11.4.*

Table 11.4: Generators for the Lucas family of dimension $f = 4m - 1$ for any $m \geq 1$

Subscript class	Generator $i = 1, \dots, m$	In terms of Lucas polynomials	As a power of g_2
3	g_{4i-1}	$(L_{(f+4i-3)/2} + L_{(f-4i+3)/2})/2$	g_2^{f-4i+3}
0	g_{4i}	$(L_{(f+4i-1)/2} - L_{(f-4i+1)/2})/2$	g_2^{f-4i+1}
1	g_{4i+1}	$(L_{(f+4i-1)/2} + L_{(f-4i+1)/2})/2$	g_2^{4i-1}
2	g_{4i+2}	$(L_{(f+4i+1)/2} - L_{(f-4i-1)/2})/2$	$-g_2^{4i+1}$

Theorem 11.5. *The even-degree Lucas circulant graph families of dimension $f \equiv 3 \pmod{4}$ have multiplicative generating set $\{1, g_2, g_2^2, \dots, g_2^{f-1}\}$.*

We now establish the arc-transitivity of even-degree Lucas circulant graphs.

Arc-transitivity of odd-degree Lucas graphs is similarly proved.

Theorem 11.6. *For any odd dimension f , the even-degree Lucas circulant graph family is arc-transitive. Denoting its order by n , its automorphism group has order $2nf$ or a multiple thereof.*

Proof. Any circulant graph on n vertices has rotational and reflective symmetries, so that its automorphism group is either the dihedral group on n elements, D_n of order $2n$, or contains the dihedral group as a subgroup. By Theorems 11.3 and 11.5, for any even-degree Lucas circulant graph of dimension f , its generating set

$G = \{1, g_2, g_2^2, \dots, g_2^{f-1}\}$ with $|g_2^f| = 1$ (all mod n), so that G is the multiplicative orbit of a single generator. This creates an additional set of symmetries, of size f , mapping any edge incident to an arbitrary vertex to any other incident edge.

Consequently, these Lucas circulant graphs are edge-transitive. As any circulant has reflexive symmetry, edge-transitivity implies arc-transitivity. \square

As far as checked, up to dimension $f = 21$, the lattice generator matrix (LGM) of an odd-degree Lucas circulant graph family has a regular format; also the LGM odd basis for even-degree Lucas families. This is in canonical form for a quasimaximal graph family. Its rows, along with the involutory vector \mathbf{v}_m , equal to half the column totals, constitute the vectors generating the corresponding lattice in \mathbb{Z}^f , by which the existence of each family may be proved for arbitrary diameter within the class.

Theorem 11.7. *The lattice generator matrix (LGM) M_f of an odd-degree Lucas circulant family of dimension f is an $f \times f$ matrix with the following form:*

$$M_f = \begin{pmatrix} 2a & -1 & 0 & \cdots & 0 & -1 \\ 1 & 2a & -1 & \ddots & & 0 \\ 0 & 1 & 2a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 2a & -1 \\ 1 & 0 & \cdots & 0 & 1 & 2a \end{pmatrix}.$$

For each odd dimension f , M_f is also the LGM odd basis for the corresponding even-degree family, consistent with the fact that even- and odd-degree families are translations of each other.

The existence of these families for all applicable degrees and corresponding diameter classes has not yet been proved. However, specific graphs of even degree up to 42 and odd degree up to 35 have been investigated by computer calculation and all have been verified, and up to these degrees the existence of the families for all higher diameters in their diameter classes is confirmed by the Existence Proof Theorem 6.16. Those of degree above 20 are new largest-known arc-transitive circulant graphs. The order of these graphs are listed below in Table 11.5. Their generating sets are simply determined from the Lucas polynomial formulae.

Table 11.5: New largest-known arc-transitive circulant graphs from Lucas polynomials

Dimension	Degree	Diameter	Order
11	22	5	8,119
		16	243,289,797
	23	11	7,881,197
13	26	22	10,161,155,672
		6	47,321
	27	19	9,240,222,891
15	30	13	141,582,068
		7	275,807
	31	15	2,537,720,636
17	34	8	1,607,521
	35	17	45,537,549,124
19	38	9	9,369,319
21	42	10	54,608,393

OTHER CIRCULANT GRAPH FAMILIES BEYOND DEGREE 20

Chapter 12 describes some series of circulant graph families beyond degree 20, created by extending sets of LGMs with common formats to higher dimensions. These families are conjectured to exist for all dimensions.

In the previous chapter, we saw how Lucas polynomials can be used to define an infinite sequence of graph families of arbitrary odd dimension. These Lucas circulant graph families are not only defined by a regular pattern of Lucas polynomials, but may also be defined by a regular pattern of lattice generator matrices to generate graph families of arbitrary odd dimension. This is facilitated by the fact that each degree is associated with its principal diameter class: $k \equiv 0 \pmod{f}$ for even degree and $k \equiv (f-1)/2 \pmod{f}$ for odd degree. For quasimaximal Abelian Cayley graph families, these are precisely the diameter classes that admit self-conjugation, as it is possible for the LGM (LGM odd basis for even degree) to have a trace of $2fa$ where $a = (2k+c)/f$ for appropriate constant c , so that the second coefficient in the polynomial in $2a$ for the order of the graphs is zero.

For the principal diameter class of each degree, in addition to the Lucas graph family sequences, there are other LGM sequences defined by regular patterns that also generate valid quasimaximal graphs for low diameters and are conjectured to extend indefinitely. Writing the order polynomial in $2a$ of a quasimaximal graph family of dimension f and odd degree as $(c_f \ c_{f-1} \dots c_0)$, then for the principal diameter class $k \equiv 0 \pmod{f}$ and setting $a = 2k/f$, we have $c_f = 1$ and $c_{f-1} = 0$. Therefore, the relative order of quasimaximal graph families is determined firstly by the value of the third coefficient, c_{f-2} .

Reprising the LGMs of the Lucas families, for odd dimension f , we have

$$M_f = \begin{pmatrix} 2a & -1 & 0 & \cdots & 0 & -1 \\ 1 & 2a & -1 & \ddots & & 0 \\ 0 & 1 & 2a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 2a & -1 \\ 1 & 0 & \cdots & 0 & 1 & 2a \end{pmatrix} \text{ and } c_{f-2} = f.$$

Therefore, the aim is to find regular sequences of LGMs giving order polynomials with third coefficient, $c_{f-2} > f$. For $f \geq 2$, we will define a sequence of $f \times f$ matrices A_f with larger c_{f-2} . First, for $f \geq 4$ and $4 \leq m \leq f$, we define $a_f(m)$ to be the $f \times f$

matrix comprised of a 3×3 block $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{bmatrix}$ with upper left element at position

$m - 2$ on the leading diagonal and zeroes elsewhere.

We also similarly define $a_f(3)$ and $a_f(2)$ to be the $f \times f$ matrices with blocks

$$\begin{bmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{bmatrix} \text{ and } \begin{bmatrix} 2a+1 & -1 \\ 1 & 2a-1 \end{bmatrix} \text{ in the upper left corner and zeroes}$$

elsewhere. For dimension $f \geq 2$, define $A_f = \begin{cases} \sum_{i=1}^{f/2} a_f(2i) & \text{for even } f \\ \sum_{i=1}^{(f-1)/2} a_f(2i+1) & \text{for odd } f. \end{cases}$

So the first four members of the sequence are

$$A_2 = \begin{pmatrix} 2a+1 & -1 \\ 1 & 2a-1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2a+1 & -1 & 0 & 0 \\ 1 & 2a-1 & -1 & -1 \\ 0 & 1 & 2a & -1 \\ 0 & 1 & 1 & 2a \end{pmatrix}, \quad A_5 = \begin{pmatrix} 2a & -1 & -1 & 0 & 0 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a & -1 & -1 \\ 0 & 0 & 1 & 2a & -1 \\ 0 & 0 & 1 & 1 & 2a \end{pmatrix}.$$

Expressed as polynomials in $2a$, these have determinants $(1 \ 0 \ 0)$, $(1 \ 0 \ 3 \ 0)$, $(1 \ 0 \ 3 \ 2 \ 0)$, $(1 \ 0 \ 6 \ 0 \ 5 \ 0)$ respectively. For even dimension f , the third coefficient $c_{f-2} = 3(f-2)/2$, and for odd dimension, $c_{f-2} = 3(f-1)/2$, which are both larger than the Lucas families' value of $c_{f-2} = f$. For dimension $2 \leq f \leq 5$, these matrices are the LGMs of the extremal and largest-known circulant graph families of odd

degree $d = 2f + 1$ and principal diameter class. These results are summarised in the following theorem and conjecture.

Theorem 12.1. *For dimension f , $2 \leq f \leq 13$, the matrix A_f defined above is the lattice generator matrix for a quasimaximal Abelian Cayley graph family of degree $2f + 1$ and principal diameter class, diameter $k \equiv 0 \pmod{f}$. Expressed as polynomials in $2a$ where $a = 2k/f$, the order polynomial has third coefficient:*

$$c_{f-2} = \begin{cases} 3(f-2)/2 & \text{for even } f \\ 3(f-1)/2 & \text{for odd } f. \end{cases}$$

For odd dimension f , A_f is also the LGM odd basis of the even-degree translate of the odd-degree graph family, for the principal diameter class, $k \equiv (f-1)/2 \pmod{f}$.

Proof. Up to dimension 13, the existence of the graphs in these families has been verified by computer program for low diameters, see Table 12.1. Therefore, by the Existence Proof Theorem 6.16, these graphs exist for all diameters in their diameter classes. □

Table 12.1: Verified graphs from LGMs with format A_f

Dim f	Degree	Diameter class, $0 \bmod \dots$	Order polynomial in $2a$ for $a = 2k/f$	Verified up to diameter
2	5	2	(1 0 0)	all
3	7	3	(1 0 3 0)	all
4	9	2	(1 0 3 2 0)	all
5	11	5	(1 0 6 0 5 0)	all
6	13	3	(1 0 6 2 5 4 0)	45
7	15	7	(1 0 9 0 19 0 7 0)	56
8	17	4	(1 0 9 2 19 10 7 6 0)	24
9	19	9	(1 0 12 0 42 0 44 0 9 0)	18
10	21	5	(1 0 12 2 42 16 44 28 9 8 0)	15
11	23	11	(1 0 15 0 74 0 138 0 85 0 11 0)	11
12	25	6	(1 0 15 2 74 22 138 68 85 60 11 10 0)	12
13	27	13	(1 0 18 0 115 0 316 0 363 0 146 0 13 0)	13

Conjecture 12.2. *Theorem 12.1 is valid for all dimensions $f \geq 2$.*

For degrees 21 and 27, these are largest-known circulant graphs, also for degree 26 for diameter greater than 6. They are included in Appendix D Table D.20 as members of families F21:0, F26:6 and F27:0. It is conjectured that for any dimension $f \geq 2$, these graph families exist for all the diameters within their diameter class. For dimensions 6 to 9, these families are not largest known, and they are conjectured not to be extremal for any dimension $f \geq 6$.

A similar sequence of odd-degree LGMs B_f can be constructed for any dimension $f \geq 4$ by pentadiagonal matrices, where the elements in the leading diagonal are $2a$ with the exception of the second element, which is $2a + 1$, and the penultimate, $2a - 1$. In the first two upper diagonals, the elements are all -1 , and 1 in the first two lower diagonals, with zeroes elsewhere. So the first two members of the sequence are

$$B_4 = \begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & -1 \\ 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 2a \end{pmatrix}, B_5 = \begin{pmatrix} 2a & -1 & -1 & 0 & 0 \\ 1 & 2a+1 & -1 & -1 & 0 \\ 1 & 1 & 2a & -1 & -1 \\ 0 & 1 & 1 & 2a-1 & -1 \\ 0 & 0 & 1 & 1 & 2a \end{pmatrix}.$$

Expressed as polynomials in $2a$, these have determinants $(1 \ 0 \ 4 \ 0 \ 0)$ and $(1 \ 0 \ 6 \ 0 \ 1 \ 0)$. For any dimension f , the third coefficient $c_{f-2} = 2(f-2)$, which is larger than the previous sequence for $f \geq 6$. For dimension 4, degree 9, these matrices are the LGMs of the largest-known Abelian Cayley graph family for the principal diameter class, $k \equiv 0 \pmod{2}$. These graphs are not circulant, but instead have cyclic rank 2, and are larger than the corresponding largest-known degree 9 circulant graph family.

Theorem 12.3. *For dimension f , $4 \leq f \leq 14$, the matrix B_f defined above is the lattice generator matrix for a quasimaximal Abelian Cayley graph family of degree $2f + 1$ and principal diameter class, diameter $k \equiv 0 \pmod{f}$. Expressed as polynomials in $2a$ where $a = 2k/f$, the order polynomial has third coefficient:*

$$c_{f-2} = 2(f-2).$$

For odd dimension f , B_f is also the LGM odd basis of the even-degree translate of the odd-degree graph family, for the principal diameter class $k \equiv (f-1)/2 \pmod{f}$.

Proof. Up to dimension 13, the existence of the graphs in these families has been verified by computer program for low diameters, see Table 12.2. Therefore, by the Existence Proof Theorem 6.16, these graphs exist for all diameters in their diameter classes. □

Conjecture 12.4. *Theorem 12.3 is valid for all dimensions $f \geq 4$.*

For dimension $f \equiv 0$ and $2 \pmod{3}$, the families are circulant, whereas for $f \equiv 1 \pmod{3}$ they are Abelian Cayley with cyclic rank 2.

For degree 21 and above, these graph families are largest known for their diameter classes, although not conjectured to be extremal. For each degree, the diameters

Table 12.2: Confirmed graphs from LGMs with format B_f

Dim f	Degree	Diameter class, 0 mod ...	Order polynomial in $2a$ for $a = 2k/f$	Cyclic rank	Verified up to diameter
4	9	2	$(1\ 0\ 4\ 0\ 0)$ $= (1\ 0\ 4\ 0)/2 \times (2\ 0)$	2	all
5	11	5	$(1\ 0\ 6\ 0\ 1\ 0)$	1	all
6	13	3	$(1\ 0\ 8\ 0\ 7\ 0\ 0)$	1	45
7	15	7	$(1\ 0\ 10\ 0\ 16\ 0\ 0\ 0)$ $= (1\ 0\ 10\ 0\ 16\ 0\ 0) \times (1\ 0)$	2	35
8	17	4	$(1\ 0\ 12\ 0\ 29\ 0\ 6\ 0\ 0)$	1	24
9	19	9	$(1\ 0\ 14\ 0\ 46\ 0\ 25\ 0\ 3\ 0)$	1	18
10	21	5	$(1\ 0\ 16\ 0\ 67\ 0\ 64\ 0\ 8\ 0\ 0)$ $= (1\ 0\ 16\ 0\ 67\ 0\ 64\ 0\ 8\ 0)/2 \times (2\ 0)$	2	15
11	23	11	$(1\ 0\ 18\ 0\ 92\ 0\ 131\ 0\ 33\ 0\ 1\ 0)$	1	11
12	25	6	$(1\ 0\ 20\ 0\ 121\ 0\ 234\ 0\ 107\ 0\ 14\ 0\ 0)$	1	12
13	27	13	$(1\ 0\ 22\ 0\ 154\ 0\ 381\ 0\ 274\ 0\ 52\ 0\ 0\ 0)$	-	-
14	29	7	$(1\ 0\ 24\ 0\ 191\ 0\ 580\ 0\ 594\ 0\ 172\ 0\ 11\ 0\ 0)$	1	14

checked produced a valid Abelian Cayley graph with two exceptions. For degree 14, the diameter 3 graph is circulant, the graphs at diameter 17 and 24 are Abelian Cayley with cyclic rank 2, and at diameter 10 no graph was found. It is possible that the diameter 10 graph exists with a higher cyclic rank. The diameter 3 graph is circulant because the value of the second cyclic order happens to be 1 in this case. The second exception is for degree 27. No solution was found for diameter 13, the only value that was small enough to be checked by computer. However, its even-translate family, with degree 26, has a confirmed member at diameter 6, a circulant graph for the same reason as above.

For degrees 22, 23, 25 and 29, these are largest-known circulant graphs. They are included in Appendix D Table D.20 as members of families F22:5, F23:0, F25:0 and F29:0. For degree 21, these graphs are largest-known non-circulant Abelian Cayley graphs. They are included in Appendix F Table F.12 as members of family A21:0. As mentioned, the only member confirmed for the Abelian Cayley graph family of degree 26, A26:6, is the smallest member with diameter 6 and is, by exception circulant. It is included in both Appendices D Table D.20 and F Table F.13.

EXTENSION OF THE EXTREMAL ORDER CONJECTURE

In Chapter 13, the Extremal Order Conjecture is extended to the third coefficient in the order polynomial. A conjecture is also discussed that all extremal Abelian Cayley graphs above threshold diameters are members of quasimaximal families. Some established theorems from the literature on asymptotically low-density lattice coverings are considered to investigate whether they might indicate the existence, for sufficiently large dimension, of extremal Abelian Cayley graph families of order greater than determined by the Extremal Order Conjecture. However, the validity of these theorems is questioned, and the conjecture is considered to remain valid.

13.1 Including bounds on the third coefficient

The Extremal Order Conjecture 3.1, in Chapter 3, states that for an extremal Abelian Cayley graph family \mathcal{A} of degree d , and corresponding dimension f , the order n of any graph of diameter k in the family is given by

$$n = \begin{cases} \frac{1}{2} \left(\frac{4}{f}\right)^f k^f + \left(\frac{4}{f}\right)^{f-1} k^{f-1} + O(k^{f-2}) & \text{for even } d \\ \left(\frac{4}{f}\right)^f k^f + O(k^{f-2}) & \text{for odd } d. \end{cases}$$

Presented as polynomials in $2a$ in vector notation, with $a = 2k/f$, we have

$$n = \begin{cases} (1 \ 2 \ c_{f-2} \ \dots \ c_0)/2 & \text{for even } d \\ (1 \ 0 \ c_{f-2} \ \dots \ c_0) & \text{for odd } d. \end{cases}$$

However, in this format, the c_i are generally not integral, unless $k \equiv 0 \pmod{f}$.

Using, instead, the substitution $a = (2k + c)/f$ where c is chosen such that a remains integral for all k in the diameter class of the family we have

$$n = \begin{cases} (1 \ c_{f-1} \ c_{f-2} \ \dots \ c_0)/2 & \text{for even } d \\ (1 \ c_{f-1} \ c_{f-2} \ \dots \ c_0) & \text{for odd } d. \end{cases}$$

In this case, the coefficient c_{f-1} is not necessarily 2 or 0 respectively, but the c_i are all integral. In Section 5.3, Table 5.11, there are examples of graph families where the second coefficient is reduced by 1, 2, 3 and 4. This is the quasimaximal defect of the family. It is conjectured that, for all degrees, families exist with arbitrary quasimaximal defect, always integral. In Chapter 9, we have seen that all known extremal and largest-known bipartite circulant graph families have quasimaximal defect of 2, and this is reflected in the Extremal Order Conjecture for Bipartite Circulant Graphs, Conjecture 9.6.

We know that the order of a graph family is equal to the magnitude of the determinant of its lattice generator matrix. In the following, we will initially consider LGMs in canonical format for odd-degree families. Let M be the LGM in canonical format of an odd-degree graph family \mathcal{A} of dimension f , diameter class $k \equiv 0 \pmod{f}$, with $a = 2k/f$, so that

$$M = \begin{pmatrix} 2a + b_1 & c_{1,2} & \dots & c_{1,f} \\ c_{2,1} & 2a + b_2 & \dots & c_{2,f} \\ \vdots & \vdots & \ddots & \vdots \\ c_{f,1} & c_{f,2} & \dots & 2a + b_f \end{pmatrix}.$$

Then order

$$\begin{aligned} n = \det(M) &= (2a)^f + \sum b_i (2a)^{f-1} + \left(\sum_{i < j} b_i b_j - \sum_{i < j} c_{ij} c_{ji} \right) (2a)^{f-2} + O((2a)^{f-3}) \\ &= \left(1 \sum b_i \sum_{i < j} b_i b_j - \sum_{i < j} c_{ij} c_{ji} \dots \right). \end{aligned}$$

Note that the third coefficient depends on the products of pairs of b_i in the leading diagonal. For any given sum $\sum b_i$, this sum of products is maximal when the b_i are chosen such that $\max_{i < j} |b_i - b_j|$ is minimised (either 0 or 1).

Now we let \mathcal{A} be quasimaximal, so that $\sum b_i = 0$. This implies that $\sum_{i < j} b_i b_j \leq 0$, with equality if and only if every $b_i = 0$. Also, for a quasimaximal family, the value of each c_{ij} is either 0, 1 or -1 , and the value of the product of each transpose pair $c_{ij} c_{ji} = 0$ or -1 . There are $f(f-1)/2$ such pairs. So an upper bound for the third coefficient of the order polynomial in $2a$ for the graph family is $f(f-1)/2$. For odd dimension f , this maximum is consistent with the canonical LGM format. However, for even dimension, the column totals would not be even as required. In this case, each column must include one zero element, giving a maximum of $f(f-2)/2$.

The previous calculations were for the diameter class $k \equiv 0 \pmod{f}$. We now generalise this to cover all diameter classes k^* , where $0 \leq k^* < f$. For the standard substitution $a = (2k + c)/f$, we need to choose an integer constant c with $-f/2 \leq c < f/2$ such that a is integral for all diameters k in the class. This is achieved by setting $c = (2(f - k^*) + \lfloor f/2 \rfloor) \pmod{f} - \lfloor f/2 \rfloor$. As $a = (2k + c)/f$, we have $4k/f = 2a - 2c/f$. For dimension f , the quasimaximal order n is given by

$$\begin{aligned} n &= (4k/f)^f + O((4k/f)^{f-2}) \\ &= (2a - 2c/f)^f + O((2a)^{f-2}) \\ &= (2a)^f - 2c(2a)^{f-1} + O((2a)^{f-2}). \end{aligned}$$

Hence $\sum b_i = -2c$. As mentioned earlier, the distribution of the b_i down the leading diagonal has an impact on the third coefficient. For a maximal solution, it is required that the value of $\sum b_i$ is distributed across the b_i as evenly as possible. Thus, the diagonal is comprised of $|2c|$ elements with value $2a \pm 1$ (depending on the sign of c) and $f - |2c|$ elements with value $2a$. Then $\sum_{i < j} b_i b_j = |2c||2c - 1|/2 = 2c^2 - |c|$. If f is odd, then a column with diagonal element $2a$ will have even sum if all the other elements are ± 1 . Similarly, for a column with diagonal element $2a \pm 1$ if f is even. Conversely, if f is odd, then a column with diagonal element $2a \pm 1$ will need to include at least one zero element for an even sum, also for any column with diagonal element $2a$ if f is even.

Thus, to ensure that all column sums are even, a parity correction is applied to the third coefficient upper bound equal to the minimum number of zeroes in the upper triangle of the LGM, being half the number of columns to be corrected. Denoting the order polynomial in $2a$ by $(c_f \ c_{f-1} \ c_{f-2} \ \dots \ c_0)$, we have:

$$c_f = 1, \ c_{f-1} = -2c, \ c_{f-2} \leq 2c^2 - |c| + f(f-1)/2 - P_o,$$

where P_o is the odd-degree parity correction given by

$$P_o = \begin{cases} |c| & \text{for odd } f \\ f/2 - |c| & \text{for even } f. \end{cases}$$

The first two coefficients, c_f and c_{f-1} are precisely the original Extremal Order Conjecture for odd degree. The third, c_{f-2} , extends the conjecture to an upper bound for the third coefficient.

For even degree, we use the LGM odd basis, which is in the same format, dividing its determinant by 2 for the order. A similar analysis leads to the following result for

quasimaximal even-degree Abelian Cayley graph families. Denoting the order polynomial in $2a$ by $(c_f \ c_{f-1} \ c_{f-2} \ \dots \ c_0)/2$, we have:

$$c_f = 1, \ c_{f-1} = 2 - 2c, \ c_{f-2} \leq 2(c-1)^2 - |c-1| + f(f-1)/2 - P_e,$$

where P_e is the even-degree parity correction given by

$$P_e = \begin{cases} |c-1| & \text{for odd } f \\ f/2 - |c-1| & \text{for even } f. \end{cases}$$

The first two coefficients, c_f and c_{f-1} are the original Extremal Order Conjecture for even degree. The third, c_{f-2} , extends the conjecture to an upper bound for the third coefficient.

The upper bound for c_{f-2} is attained when the number of off-diagonal zeroes in the LGM is minimised. Thus, we have the following conjecture for the order of extremal circulant and Abelian Cayley graphs.

Conjecture 13.1. *Extended Extremal Order Conjecture for Abelian Cayley graphs.*

The order of an extremal Abelian Cayley graph of degree d and diameter k is denoted $Ext_{AbCay}(d, k)$, and similarly $Ext_{circ}(d, k)$ for an extremal circulant graph. To convert the order polynomials from diameter k , in arbitrary diameter class k^ , to parameter $2a$, we use the standard substitution $a = (2k + c)/f$ where $c = (2(f - k^*) + \lfloor f/2 \rfloor) \bmod f - \lfloor f/2 \rfloor$. For any diameter $k > k_d$ for some threshold k_d depending on d :*

For even degree d , $Ext_{AbCay}(d, k) = (1 \ 2(1-c) \ c_{f-2} \ c_{f-3} \ \dots)/2$ where

$$0 \leq c_{f-2} \leq \begin{cases} 2(c-1)^2 + f(f-2)/2 & \text{for even } f \\ 2(c-1)^2 - 2|c-1| + f(f-1)/2 & \text{for odd } f. \end{cases}$$

For odd degree d , $Ext_{AbCay}(d, k) = (1 \ -2c \ c_{f-2} \ c_{f-3} \ \dots)$ where

$$0 \leq c_{f-2} \leq \begin{cases} 2c^2 + f(f-2)/2 & \text{for even } f \\ 2c^2 - 2|c| + f(f-1)/2 & \text{for odd } f. \end{cases}$$

These bounds are identical for $Ext_{circ}(d, k)$.

It is interesting to compare largest-known circulant and Abelian Cayley graph families of dimension 3 and above with newly conjectured third coefficient upper bound. For dimension 3, the largest-known circulant graph families of degrees 6 and 7 achieve this upper bound for all diameter classes. For dimension 4, the largest-known circulant graph families of degree 8 also achieve the upper bound, and the degree 9 Abelian Cayley graph family for one of the diameter classes. For dimension 5, none of the circulant families achieve the upper bound. However, the Abelian Cayley graph

families of both degrees 10 and 11 achieve the upper bound for two of the five diameter classes. Beyond dimension 5, none of the largest-known Abelian Cayley graph families achieve the conjectured third coefficient upper bound. See Tables 13.1 and 13.2.

To reconfirm, the specific values for the first two coefficients and the upper bound for the third in the extended Extremal Order Conjecture for circulant and non-circulant Abelian Cayley graph families are a direct consequence of the conjecture that their LGMs are quasimaximal and canonical. Over the range analysed, it appears that the third coefficient of largest-known circulant graph families achieves roughly 70% of the upper bound, while for Abelian Cayley graph families it is around 80%. There is too little data for a reasonable conjecture as to how these proportions evolve for higher degrees.

For any dimension and diameter class, there are only finitely many combinations of values in the leading diagonal of a canonical odd-degree quasimaximal LGM that admit a third coefficient in the order polynomial above any given value. By checking and discounting all of these for a largest-known circulant or Abelian Cayley graph family, it is possible to confirm that the graph family is extremal within the context of families that are quasimaximal. This has been checked for all degrees up to 13.

Those families confirmed to be extremal quasimaximal are indicated by bold text in the final two columns of Tables 13.1 and 13.2.

Table 13.1: Comparison of largest-known circulant and Abelian Cayley graph families' third coefficients with the conjectured upper bound: Even degrees

Dimension f	Degree d	Diameter class k (mod f)	Constant c in $a = (2k + c)/f$	Order polynomials in $2a$ (1)		Largest-known families' third coefficient (2)	
				Conjectured upper bound	First three coeffs	Circulant	Abelian Cayley
3	6	0	0	(1 2 3 ...		3	3
3	6	1	1	(1 0 3 ...		3	3
3	6	2	2	(1 -2 3 ...		3	3
4	8	0	0	(1 2 6 ...		6	6
4	8	1	2	(1 -2 6 ...		6	6
5	10	0	0	(1 2 10 ...		8	8
5	10	1	3	(1 -4 14 ...		13	14
5	10	2	1	(1 0 10 ...		6	7
5	10	3	-1	(1 4 14 ...		13	14
5	10	4	2	(1 -2 10 ...		8	8
6	12	0	0	(1 2 14 ...		11	12
6	12	1	-2	(1 6 30 ...		24	25
6	12	2	2	(1 -2 14 ...		11	12
7	14	0	0	(1 2 21 ...		15	16
7	14	1	-2	(1 6 33 ...		28	30
7	14	2	3	(1 -4 25 ...		21	22
7	14	3	1	(1 0 21 ...		14	13
7	14	4	-1	(1 4 25 ...		21	22
7	14	5	4	(1 -6 33 ...		28	30
7	14	6	2	(1 -2 21 ...		15	16
8	16	0	0	(1 2 26 ...		20	
8	16	1	-2	(1 6 42 ...		33	
8	16	2	4	(1 -6 42 ...		33	
8	16	3	2	(1 -2 26 ...		20	
9	18	0	0	(1 2 36 ...		23	
9	18	1	-2	(1 6 48 ...		37	
9	18	2	5	(1 -8 60 ...		50	
9	18	3	3	(1 -4 40 ...		29	
9	18	4	1	(1 0 36 ...		20	
9	18	5	-1	(1 4 40 ...		29	
9	18	6	-3	(1 8 60 ...		50	
9	18	7	4	(1 -6 48 ...		37	
9	18	8	2	(1 -2 36 ...		23	
10	20	0	0	(1 2 42 ...		26	
10	20	1	-2	(1 6 58 ...		42	
10	20	2	-4	(1 10 90 ...		70	
10	20	3	4	(1 -6 58 ...		42	
10	20	4	2	(1 -2 42 ...		26	

(1) The order polynomial in $2a$ is divided by 2

(2) Extremal quasimaximal shown in bold text

Table 13.2: Comparison of largest-known circulant and Abelian Cayley graph families' third coefficients with the conjectured upper bound: Odd degrees

Dimension f	Degree d	Diameter class k (mod f)	Constant c in $a = (2k + c)/f$	Order polynomials in $2a$			Largest-known families'	
				Conjectured upper bound	third coefficient (1)			
				First three coeffs	Circulant	Abelian	Cayley	
3	7	0	0	(1 0 3 ...	3		3	
3	7	1	1	(1 -2 3 ...	3		3	
3	7	2	-1	(1 2 3 ...	3		3	
4	9	0	0	(1 0 4 ...	3		4	
4	9	1	2	(1 -4 12 ...	9		10	
5	11	0	0	(1 0 10 ...	6		7	
5	11	1	-2	(1 4 14 ...	12		14	
5	11	2	1	(1 -2 10 ...	8		8	
5	11	3	-1	(1 2 10 ...	8		8	
5	11	4	2	(1 -4 14 ...	12		14	
6	13	0	0	(1 0 12 ...	8		12	
6	13	1	-2	(1 4 20 ...	16		17	
6	13	2	2	(1 -4 20 ...	16		17	
7	15	0	0	(1 0 21 ...	14		15	
7	15	1	-2	(1 4 25 ...	20		22	
7	15	2	3	(1 -6 33 ...	28		30	
7	15	3	1	(1 -2 21 ...	14		16	
7	15	4	-1	(1 2 21 ...	14		16	
7	15	5	-3	(1 6 33 ...	28		30	
7	15	6	2	(1 -4 25 ...	20		22	
8	17	0	0	(1 0 24 ...	17			
8	17	1	-2	(1 4 32 ...	25			
8	17	2	-4	(1 8 56 ...	44			
8	17	3	2	(1 -4 32 ...	25			
9	19	0	0	(1 0 36 ...	20		27	
9	19	1	-2	(1 4 40 ...	27			
9	19	2	-4	(1 8 60 ...	50			
9	19	3	3	(1 -6 48 ...	36			
9	19	4	1	(1 -2 36 ...	22			
9	19	5	-1	(1 2 36 ...	22			
9	19	6	-3	(1 6 48 ...	36			
9	19	7	4	(1 -8 60 ...	50			
9	19	8	2	(1 -4 40 ...	27			

(1) Extremal quasimaximal shown in bold text

13.2 Why the conjectured third coefficient upper bound is not achieved

Beyond dimension 6, no largest-known graph family achieves the third coefficient upper bound. From the formulae for the third coefficient of the polynomial for the determinant of an odd-degree canonical quasimaximal LGM, it is evident that the value is maximised only when the number of zero elements is minimised. As far as checked, as the dimension increases, the minimum number of zero elements in the LGM of a graph family also increases. So not all matrices in canonical LGM format

are LGMs for graph families. It appears that the range of lattices generated by such matrices, together with the associated set of Lee spheres, span the critical zone around extremality. That is, they generate lattice coverings of space that are suboptimal (with redundant overlap between Lee spheres) and also optimal (with minimal overlap, or equivalently minimum covering density). But they also generate lattices that fall the other side of the limit, so that the Lee spheres do not achieve a covering but leave some space uncovered between the spheres. This would explain why canonical LGMs do not all generate graph families and why the third coefficient is not achieved.

The simplest example of a matrix in odd-degree canonical quasimaximal LGM format that does not generate a family has dimension 4, and was also given in Section 5.3. Consider the matrix M as a candidate LGM for a graph family of degree 9 with $a = k/2$, diameter class $k \equiv 0$, and

$$M = \begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & 1 \\ 0 & 1 & -1 & 2a \end{pmatrix}.$$

This has determinant polynomial $(1 \ 0 \ 4 \ 0 \ 4)$, which exceeds the largest-known circulant family, with order $(1 \ 0 \ 3 \ 2 \ 0)$, and the largest-known Abelian Cayley family, with order $(1 \ 0 \ 4 \ 0 \ 0)$.

The reason M is not the LGM for a graph family is that its lattice, with associated Lee spheres, leaves some points of space uncovered. Specifically, there are two points, $(1 \ a \ a \ 0)$ and $(a \ 0 \ -1 \ a)$, that are not within a distance $2a = k$ of any of the vertices defined by the four lattice vectors (rows of the matrix), nor within $k - 1$ of the involutory vector $(a + 1 \ a \ a - 1 \ a)$. This arises from the fact that the distance between the first and fourth vector, and between the second and third, is $4a + 2 = 2k + 2$ in each case, leaving the uncovered points at the corresponding midpoints, at a distance of $k + 1$ from either vertex.

For any dimension, as the number of zero elements in a candidate LGM decreases, then the distances between the lattice points increase, increasing the likelihood that the covering fails.

13.3 Potentially sporadic graphs and diameter thresholds

Largest-known Abelian Cayley graphs (circulant and non-circulant) of arbitrary degree and any diameter beyond a low threshold are all members of largest-known quasimaximal graph families with canonical LGMs. An important question is whether this holds true for extremal Abelian Cayley graphs of arbitrary degree, or whether there exist extremal sporadic graphs that are not members of a graph family. If such sporadic graphs exist, then discovering the extremal graph families would not necessarily reveal the extremal graphs. Also, upper bounds on the order of extremal graphs that were based on the structure of graph families would not necessarily be valid. In particular, the Extremal Order Conjecture 3.1 would not be valid universally.

None of the known extremal circulant graphs below the diameter thresholds is a member of a quasimaximal graph family. If it emerges that they are sporadic, then this would prove that sporadic graphs exist and support a view that, at some diameters above the thresholds, extremal graphs may also be sporadic. However, if they are all members of subquasimaximal families, then this would support the opposite view, that sporadic graphs do not exist and that all extremal graphs above the diameter threshold belong to extremal graph families. The known extremal potentially sporadic graphs up to degree 9 are presented in Table 13.3.

Table 13.3: Extremal potentially sporadic circulant graphs up to degree 9

Degree	Diameter	Graph in largest-known family		Extremal potentially sporadic graph		
		Order	Family	Order	Family	Generating set
7	2	24	F7:2A	26	G7:2A	1, 2, 8
			F7:2B		G7:2B	1, 3, 8
8	2	32	F8:0	35	G8:2A	1, 6, 7, 10
					G8:2B	1, 7, 11, 16
9	2	32	F9:0	42	G9:2A	1, 5, 14, 17
					G9:2B	2, 7, 8, 10
9	3	108	F9:1a F9:1b	130	G9:3A	1, 8, 14, 17
					G9:3B	1, 8, 20, 35
					G9:3C	1, 26, 49, 61
					G9:3D	2, 8, 13, 32
9	4	312	F9:0	320	G9:4	1, 15, 25, 83

For each of these 11 potentially sporadic graphs in turn, computer searches for circulant graph families that contained the graph were conducted based on LGMs of increasing quasimaximal defect. After extensive search, each of these graphs was found to belong to a subquasimaximal family, see Table 13.4. So none of them are sporadic.

Table 13.4: Subquasimaximal families containing extremal potentially sporadic graphs

Family	Order	$a =$	LGM or LGM odd basis	Order polynomial in $2a$	Quasimaximal defect
G7:2A	26	$(2k-1)/3$	$\begin{pmatrix} 2a-2 & -2 & -1 & \\ 3 & 2a & 2 & \\ -1 & -2 & 2a+1 & \end{pmatrix}$	$(1 -1 7 8)$	3
G7:2B	26	$(2k-1)/3$	$\begin{pmatrix} 2a-1 & -2 & -2 & \\ 1 & 2a-1 & -3 & \\ 2 & 1 & 2a+1 & \end{pmatrix}$	$(1 -1 8 6)$	3
G8:2A	35	$k/2$	$\begin{pmatrix} 2a-1 & -3 & -2 & 0 \\ 2 & 2a & 1 & -1 \\ 0 & -3 & 2a & 1 \\ -1 & 2 & -1 & 2a \end{pmatrix}$	$(1 -1 12 4 6)/2$	3
G8:2B	35	$k/2$	$\begin{pmatrix} 2a-1 & -1 & -1 & -1 \\ 3 & 2a & -2 & -1 \\ 3 & 1 & 2a & 0 \\ 3 & 0 & -1 & 2a \end{pmatrix}$	$(1 -1 11 8 2)/2$	3
G9:2A	42	$k/2$	$\begin{pmatrix} 2a-3 & -1 & -2 & -1 \\ 3 & 2a-2 & 0 & 0 \\ 2 & -1 & 2a+1 & 0 \\ 2 & 0 & 1 & 2a+1 \end{pmatrix}$	$(1 -3 6 11 4)$	3
G9:2B	42	$k/2$	$\begin{pmatrix} 2a-2 & -1 & -1 & -1 \\ 2 & 2a & -1 & 0 \\ 2 & 0 & 2a & -1 \\ 2 & -1 & 0 & 2a \end{pmatrix}$	$(1 -2 6 5 8)$	2
G9:3A	130	$k/2$	$\begin{pmatrix} 2a-2 & -3 & -1 & -3 \\ 2 & 2a-2 & 0 & -2 \\ 1 & -2 & 2a-1 & 0 \\ 2 & -2 & -1 & 2a \end{pmatrix}$	$(1 -5 17 9 4)$	5
G9:3B	130	$k/2$	$\begin{pmatrix} 2a-2 & -4 & -2 & -2 \\ 2 & 2a-2 & 1 & 1 \\ -1 & -3 & 2a-1 & -2 \\ 2 & -2 & 1 & 2a \end{pmatrix}$	$(1 -5 25 -13 -2)$	5
G9:3C	130	$k/2$	$\begin{pmatrix} 2a-2 & -2 & -2 & -2 \\ -1 & 2a-2 & -3 & -2 \\ 1 & 3 & 2a-1 & -1 \\ 1 & 2 & -1 & 2a \end{pmatrix}$	$(1 -5 22 -2 -8)$	5
G9:3D	130	$k/2$	$\begin{pmatrix} 2a-3 & -1 & -3 & -4 \\ 2 & 2a-2 & 1 & -1 \\ 2 & -3 & 2a-1 & -4 \\ 2 & 1 & 0 & 2a \end{pmatrix}$	$(1 -6 31 -27 13)$	6
G9:4	320	$k/2$	$\begin{pmatrix} 2a-2 & -4 & -1 & -1 \\ 3 & 2a-2 & 1 & 2 \\ -1 & -3 & 2a-1 & -1 \\ -2 & -1 & 1 & 2a \end{pmatrix}$	$(1 -5 23 2 8)$	5

Increased quasimaximal defect increases the extent of overlap of the Lee spheres in the corresponding lattice covering. This tends to increase the number of distinct subquasimaximal families that the graph is a member of. So while it is usually the case that graphs of low diameter belonging to a quasimaximal family belong to no other family, graphs belonging to a subquasimaximal family may belong to multiple families. At higher diameter, this duplication becomes increasingly rare as the asymptotics of the order polynomial of each family separate them from each other. For example, the degree 9 diameter 2 graph G9:2B was found to be a member of at least nine quasimaximal families with distinct order polynomials and quasimaximal defect ranging from 2 to 5. These results provide support for the conjecture that all Abelian Cayley graphs belong to at least one graph family.

The last question to address on this topic is why extremal graphs below the diameter threshold for each degree are not members of a quasimaximal family. Regarding the order of its graphs, a quasimaximal family primarily differs from a subquasimaximal family in the second coefficient of its order polynomial. We may consider the contribution of the first two terms in the order polynomial for a quasimaximal family (which are, by definition, determined uniquely by its degree and diameter) as a proportion of the order of the largest-known graph for any given degree and diameter.

This proportion increases with diameter but decreases with degree. If we consider a low diameter such as 3, for example, then this proportion is about 84% for degree 7, falling to 62% for degree 9, and only 1% by degree 20. It appears from the limited available data that the largest-known (quasimaximal) families dominate when the proportion is above about 60%. Below this value, the extremal graphs belong to subquasimaximal families. If this remains true in broad terms for higher degrees, then the diameter threshold, above which all extremal Abelian Cayley graphs belong to extremal families, increases without limit as the degree increases. A corollary is that many of the largest-known graphs of dimension 6 and above listed in Appendices D, E and F are almost certainly not extremal, especially those members of quasimaximal families with diameter below the threshold, see Table 13.5.

Notwithstanding this, it is conjectured that for any degree, a finite diameter threshold exists above which all extremal Abelian Cayley graphs (circulant and non-circulant) belong to an extremal quasimaximal graph family.

Conjecture 13.2. *For any degree d , there is a diameter threshold k_d such that, for any diameter $k \geq k_d$, any Abelian Cayley graph of order $Ext_{abCay}(d, k)$ belongs to an extremal quasimaximal graph family and any circulant graph of order $Ext_{circ}(d, k)$ belongs to an extremal quasimaximal graph family.*

Table 13.5: Diameter thresholds: quasimaximal families compared with 60% proportion

Degree	Lowest diameter where largest-known graph is in quasimaximal family	Proportion* at this diameter	Approximate diameter threshold for 60% proportion*
6	1	42%	1
7	3	84%	2
8	3	65%	3
9	5	89%	3
10	4	60%	4
11	5	72%	5
12	6	61%	6
13	7	70%	6
14	5	36%	8
15	5	36%	8
16	5	23%	11
17	5	24%	11
18	6	22%	13
19	6	22%	13
20	5	8%	16

*based on sum of first two quasimaximal terms as proportion of largest-known order

13.4 Investigation of higher asymptotic lower bounds

The question of the existence of lower bounds that are higher than the Extremal Order Conjecture for Abelian Cayley graphs, Conjecture 3.1, was raised in Section 2.2 and mentioned in Chapter 3. Relevant papers by Gritzmman in 1985 [19] and recently by Ordentlich, Regev and Weiss [43] build on original work by Rogers published in 1959 [46]. In this section, some of the principles supporting Rogers results are discussed. Doubts are raised about the validity of these results, and the author reconfirms their confidence in the validity of the Extremal Order Conjecture for all dimensions.

As discussed earlier, the Extremal Order Conjecture and the search described in this thesis for largest-known graph families are both based on the structure of the canonical lattice generator matrices for their corresponding coverings of \mathbb{Z}^f , where f is the dimension of the graphs. The *density* D of a lattice covering of \mathbb{Z}^f by a convex polytope of volume V_p centred on all points of a regular lattice with unit cell volume V_L , equal to the determinant of its lattice generator matrix, is given by $D = V_p/V_L$. Clearly, for a covering, $D \geq 1$. For lattice generator matrices that determine graph families, the graph order is optimised by minimising the density.

If the Golomb-Welch conjecture [17] is true, then the Abelian Cayley graph order upper bound $Upp_{AbCay}(d, k)$ is not achieved for dimension $f \geq 3$ because it is impossible to tile \mathbb{Z}^f with Lee spheres. The continuous analogue is the impossibility

of tiling f -dimensional Euclidean space \mathbb{R}^f with f -orthoptoxes, also called cross-polytopes or dual f -cubes. A simple approach to the problem would be to determine the maximum volume of a rectangular f -cuboid contained within an f -orthoptox, and then position the overlapping f -orthoptoxes so that their inscribed f -cuboids achieve a perfect tiling. Assuming the f -cuboid is aligned along the same axes as the f -orthoptox, then it is straightforward to prove that the f -cuboid with maximum f -volume contained within an f -orthoptox of radius k (the distance from its centre to a vertex), is an f -cube of radius k/\sqrt{f} and edge length $2k/f$ centred at the centre of the f -orthoptox. This f -cube has f -volume $(2/f)^f k^f$ compared with $(2^f/f!)k^f$ for the f -orthoptox, giving a density of $f^f/f!$. This is worse than the Extremal Order Conjecture by a factor of 2^{f-1} . Therefore, this simple approach does not provide better solutions.

The Extremal Order Conjecture has a corresponding lattice covering density in \mathbb{Z}^f of $f^f/(2^{f-1}f!)$. Using Stirling's approximation, this is asymptotically equivalent for large f to $\sqrt{2/(\pi f)}(e/2)^f$, or about $(0.637/\sqrt{f})(1.359)^f$. This is exponential in the dimension. However, two authors have presented much better asymptotic upper bounds for the lowest density of lattice coverings of f -dimensional Euclidean space, \mathbb{R}^f , by spheres and by convex polytopes. Rogers' 1959 paper [46] established an upper bound for spheres of $cf(\log_e f)^{(1/2)\log_2(2\pi e)}$, or about $cf(\log_e f)^{2.0471}$, where the constant c is independent of f , and for arbitrary convex polytopes, of $f^{\log_2 \log_e f + c}$. Rogers' convex-polytopes upper bound is exponential in $\log \log f$, which is a significant advance on the Extremal Order Conjecture. Gritzmman improved on this in 1985 [19] with a refinement of Rogers' sphere covering with certain hyperplane symmetry assumptions for the convex polytopes that are satisfied to an integer approximation by the Lee spheres corresponding to Abelian Cayley graphs. Gritzmman's density upper bound for this category of convex polytopes is $cf(\log_e f)^{1+\log_2 e}$, or about $cf(\log_e f)^{2.4427}$, where the constant c does not depend on f or the polytope.

Very recently, Ordentlich, Regev and Weiss [43] have published an improved upper bound, cf^2 , for lowest-density lattice coverings of \mathbb{R}^f by an arbitrary compact convex set with nonempty interior, K , where the constant c is independent of f and K . This is an improvement on Rogers' result, but for the symmetric convex bodies covered by Gritzmman, Gritzmman's result remains the best. These upper bounds on the lattice covering density are much better than the Extremal Order Conjecture gives. If true, these results would imply that, for sufficiently large dimension, the Extremal Order Conjecture would no longer be valid, and would raise the question at what dimension the Extremal Order Conjecture first failed.

Gritzmann makes the observation that Rogers' insight was to show 'that certain multiple cylinders provide a rather efficient covering of space'. Their proofs are developed by constructing a *hypercylinder* embedded within the sphere or convex polytope. For sufficiently large dimension f , they define a parameter $h = \lceil \log_2 \log_e f + 4 \rceil$ (renamed here from the original terminology, dimension n and parameter k , in order to avoid confusion). There is also a constraint that $f > h$, which implies that $h \geq 5$. The hypercylinder is defined in \mathbb{R}^f as the Cartesian product $K \times P$ of an $(f - h)$ -dimensional sphere or convex polytope K and an h -dimensional hypercube P . The efficient covering is then achieved by constructing the lattice so that the embedded hypercylinders are perfectly aligned so that their constituent hypercubes form a tiling of the corresponding h -dimensional subspace. The remaining $f - h$ dimensions are apparently covered in a more conventional way by translates of K .

In Rogers' paper there is an explicit formula for the upper bound in terms of the dimension f and parameter h . All the terms in the formula are positive for any f except for a term in square brackets, $[\frac{1}{4}(f - h) \log_e \frac{27}{16} - 3 \log_e(f - h)]$, that is positive only for $f \geq 115$. This term is also present in Gritzmann's formula for symmetric convex polytopes. For dimension $f = 115$, the resultant upper bound density by Rogers' formula is 120,670, compared with a density exceeding 10^{14} for the Extremal Order Conjecture. Apart from the term turning positive at $f = 115$, there are several steps in the derivation of the formulae that are valid only 'for sufficiently large' dimension. In this context, Gritzmann, in explaining that not many hyperplanes of symmetry are required, gives the example of needing merely nine for a dimension of 1,000,000,000! Therefore, it is quite possible that the formulae are not valid until the dimension exceeds 1,000 or more.

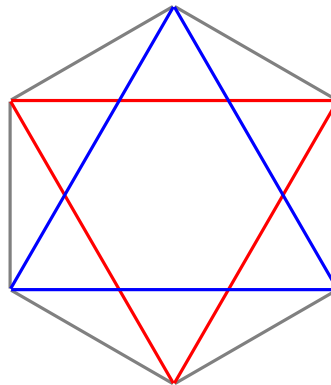
Notwithstanding the lack of clarity in the valid range of dimensions for these upper bounds, it is interesting to explore their possible relevance for low dimensions within computational range. Lee spheres are a special case of convex polytopes with rotational and reflexive symmetry in all dimensions. In addition, of course, they are integer simplexes defined in \mathbb{Z}^f rather than continuous bodies in \mathbb{R}^f . For application of the embedded hypercylinder covering approach, we initially consider the continuous analogue: f -orthoplexes of radius k within \mathbb{R}^f . For dimension 2, this is a square diamond and for dimension 3, a regular octahedron. It has f -volume $V_o = (2k)^f / f!$. Now suppose that we pick one dimension; let one of the lattice vectors be aligned along its axis and let all the other lattice vectors be orthogonal to it. We then consider the plane formed by the addition of any other lattice vector. The intersection with this plane of an f -orthoplex centred on the axis is a diamond of radius k . If we

slide two such f -orthoplexes together so that they overlap by a length $2x$ along the axis, then the maximum width of the overlap in the plane is also $2x$. In f -dimensional space the $(f-1)$ -dimensional intersection of the two f -orthoplexes will be an $(f-1)$ -orthoplex of radius x and $(f-1)$ -volume $(2x)^{f-1}/(f-1)!$. This defines the $(f-1)$ -dimensional cross-section of a hypercylinder running along the axis of the chosen dimension. Each f -orthoplex along the axis adds a length of $2k-2x$ to the hypercylinder, and therefore adds an f -volume of $2(k-x)(2x)^{f-1}/(f-1)!$. This has a maximum at $x = k(f-1)/f$, when the length of the hypercylinder segment is $2k/f$ and its cross-section has $(f-1)$ -area $\frac{2^{f-1}}{(f-1)!}(\frac{f-1}{f})^{f-1}k^{f-1}$. So its volume is $V_c = \frac{2^f}{f!}(\frac{f-1}{f})^{f-1}k^f$. Thus, the density of the hypercylinder segment in the f -orthoplex is $V_c/V_o = 2(\frac{f-1}{f})^{f-1}$.

Although these embedded hypercylinders achieve a tiling of the 1-dimensional subspace, they do not in general tile the whole space. Therefore, the volume of the hypercylinder is an upper bound for the volume of the corresponding lattice that in general will not be achieved. When the dimension is 3, the cross-section of the hypercylinder is 2-dimensional and is able to tile the subspace. So in this case the bound is achieved. The volume of the hypercylinder then has cubic coefficient $16/27$, which is only half of the $32/27$ of largest-known degree 6 graph families.

The previous construction does not take full advantage of the fact that orthoplexes have parallel opposite faces. We again consider the case of dimension 3, so that the orthoplexes are regular octahedra. We choose an axis orthogonal to the plane of a face, and stack octahedra with their centres on the axis and with the same alignment, so that faces of neighbouring octahedra are touching. Then each adjacent pair touches at the intersection of the two equilateral triangles with one inverted relative to the other, resulting in a common area in the form of a regular hexagon with area $k^2/\sqrt{3}$, see Figure 13.1.

Figure 13.1: Top view of an octahedron, with the top face shown in blue and the hidden bottom face in red defining the hexagon



This defines a hexagonal cylinder of length $2k/\sqrt{3}$ embedded in the octahedron, with a volume of $(2/3)k^3$. With appropriate alignment, these hexagons achieve a perfect tiling of the plane, so that the corresponding lattice represents an Abelian Cayley graph family of order $(2/3)k^3 + O(k^2)$. Although higher than the asymptotic order of the first construction, this is still not good enough to improve on the largest-known family with order $(32/27)k^3 + O(k^2)$.

It appears from this analysis that there is nothing to be gained by including a single cylindrical dimension in the lattice structure. Nevertheless, it demonstrates the approach that is the basis of Rogers' and Gritzmann's theorems, where a minority of the dimensions are used for the length of cylinders while most are covered in some other way.

Rogers' 1959 paper [46] was his last on the subject, but in 1964 he published a book *Packing and Covering* [47] to gather together all the known results in the theory of packing and covering in f -dimensional space for f larger than 3. In this book, Rogers notes that the result in his 1959 paper is difficult to establish and instead presents a weaker result that is 'much easier to prove'. Thus, instead of defining $h = \lceil \log_2 \log_e f + 4 \rceil$, we have $h = \lceil \log_2 f + \log_2 \log_2 f + 1 \rceil$. And instead of a lattice covering density $\theta_L(K) \leq f^{\log_2 \log_e f + c}$, we have $\theta_L(K) \leq f^{\log_2 f + c \log_2 \log_2 f}$. This result is stated on page 66 of the book as Theorem 5.8.

As in his earlier paper, the proof involves the construction of inscribed hypercylinders, where the minority, h , of the dimensions are tiled by a hypercube C , and the majority of the dimensions are covered by an $(f - h)$ -dimensional cross-section of the convex body. The proportion h/f of dimensions tiled by the hypercubes tends to zero with increasing dimension, so that the particular mechanism of this construction plays an increasingly minor role. Moreover, hypercubes are inscribed in the way described as a simple approach in the second paragraph of this section, contributing for its dimensions a significantly worse density than the Extremal Order Conjecture.

In his book [47], Rogers states that he confines his attention to very special subsets of lower-triangular unimodular matrices, asserting that, in his opinion, 'every lattice with determinant 1 can be approximated arbitrarily closely' by such matrices. Rogers defined the first variant for packings and the second, a subset of the first, for

coverings (employed in Theorem 5.5):

$$\begin{pmatrix} \chi & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \chi & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \chi & 0 \\ \alpha_1\chi & \alpha_2\chi & \cdots & \alpha_{f-2}\chi & \alpha_{f-1}\chi & \eta \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{f-2} & \alpha_{f-1} & 1 \end{pmatrix}$$

where $\eta = 1/\chi^{f-1}$ and $0 \leq \alpha_i \leq 1$.

For integer coverings, these are scaled appropriately. For each parameter set, the determinant is equal to the volume of a unit cell of the lattice and to the order of the corresponding Abelian Cayley graph. The depth (Manhattan norm) of the deep holes of the lattice is equal to the minimum radius of a covering with Lee spheres and to the diameter of the graph. For the first, more general, matrix variant, computer runs were performed on all relevant combinations of the parameters χ, η and α_i for dimensions 4 to 9 to determine the depth of the deep holes in each case, along with its determinant $\chi^{f-1}\eta$, in order to discover the maximum graph order for each diameter. In each case, the optimum occurred when all the α_i took the common value of $1/2$. The results are presented in Tables 13.6 and 13.7.

Table 13.6: Optimum matrices in Rogers' format for a range of diameters, for dimensions 4, 5 and 6

Diameter k	Dimension $f = 4$			Dimension $f = 5$			Dimension $f = 6$		
	χ	η	order	χ	η	order	χ	η	order
5	6	2	432						
6	6	4	864	6	1	1296			
7	8	3	1536	6	3	3888			
8	8	5	2560	6	5	6480	6	2	15552
9	10	4	4000	8	3	12288	6	4	31104
10	10	6	6000	8	5	20480	6	6	46656
11	12	5	8640	10	3	30000	8	3	98304
12	12	7	12096	10	5	50000	8	5	163840
13				10	7	70000	8	7	229376
14				12	5	103680	10	4	400000
15				12	7	145152	10	6	600000
16				14	5	192080	10	8	800000
17				14	7	268912	12	5	1244160
18				14	9	345744	12	7	1741824
19							12	9	2239488
20							14	6	3226944
21							14	8	4302592
22							14	10	5378240

Table 13.7: Optimum matrices in Rogers' format for a range of diameters, for dimensions 7, 8 and 9

Diameter k	Dimension $f = 7$			Dimension $f = 8$			Dimension $f = 9$		
	χ	η	order	χ	η	order	χ	η	order
9	6	1	46656						
10	6	3	139968						
11	6	5	233280	6	2	559872			
12	8	1	262144	6	4	1119744	6	1	1679616
13	8	3	786432	6	6	1679616	6	3	5038848
14	8	5	1310720	8	1	2097152	6	5	8398080
15	8	7	1835008	8	3	6291456	6	7	11757312
16	10	3	3000000	8	5	10485760	8	1	16777216
17	10	5	5000000	8	7	14680064	8	3	50331648
18	10	7	7000000	10	2	20000000	8	5	83886080
19	12	3	8957952	10	4	40000000	8	7	117440512
20	12	5	14929920	10	6	60000000	8	9	150994944
21	12	7	20901888	10	8	80000000	10	3	300000000
22	12	9	26873856	12	3	107495424	10	5	500000000
23				12	5	179159040	10	7	700000000
24				12	7	250822656	10	9	900000000
25				12	9	322486272	12	3	1289945088
26							12	5	2149908480
27							12	7	3009871872
28							12	9	3869835264
29							12	11	4729798656

For each dimension studied, the optimum matrices can be described by a single parameter a within each diameter class, in the same way as the largest-known Abelian Cayley graph families. The corresponding Abelian Cayley graphs families do not have order greater than quasimaximal. They are quasimaximal with maximum odd girth, but are not largest known. For example, for dimension 5, degree 10, there are five families, one for each diameter class modulo 5, see Table 13.8.

Table 13.8: Abelian Cayley graph families with LGM in Rogers' format for dimension 5, compared with largest known

Diameter $k \pmod{5}$	Parameter a			Cyclic rank	Order polynomial in $2a$	
		χ	η		Rogers' format	Largest known
0	$2k/5$	$2a$	$a + 1$	4	$(1\ 2\ 0\ 0\ 0\ 0)/2$	$(1\ 2\ 8\ 8\ 5\ 2)/2$
1	$(2k + 3)/5$	$2a$	$a - 2$	4	$(1\ -4\ 0\ 0\ 0\ 0)/2$	$(1\ -4\ 14\ -24\ 17\ -4)/2$
2	$(2k + 1)/5$	$2a$	a	5	$(1\ 0\ 0\ 0\ 0\ 0)/2$	$(1\ 0\ 7\ 0\ 0\ 0)/2$
3	$(2k - 1)/5$	$2a$	$a + 2$	4	$(1\ 4\ 0\ 0\ 0\ 0)/2$	$(1\ 4\ 14\ 24\ 17\ 4)/2$
4	$(2k + 2)/5$	$2a$	$a - 1$	4	$(1\ -2\ 0\ 0\ 0\ 0)/2$	$(1\ -2\ 8\ -8\ 5\ -2)/2$

In each case, in the last row of the LGM, $\alpha_i = 1/2$

These quasimaximal graph families are noteworthy in that their lattice generator matrices, being in Rogers' format, are not in canonical LGM format. However, they are isomorphic to graph families with LGMs in canonical quasimaximal format. For example, the LGM below for degree 10, diameter class 1 (mod 5) (along with its

LGM odd basis) generates a graph family isomorphic to the family generated by the Rogers LGM:

$$\begin{array}{cc} \text{Canonical LGM} & \text{Canonical LGM odd basis} \\ \left(\begin{array}{ccccc} a & -a & -a & -a & -a+2 \\ a & a & -a & -a & -a+2 \\ a & -a & a & -a & -a+2 \\ a & -a & -a & a & -a+2 \\ a & -a & -a & -a & a-2 \end{array} \right) & \left(\begin{array}{ccccc} 2a & 0 & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 & 0 \\ 0 & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 0 & 2a-4 \end{array} \right) \end{array}.$$

Note that the involutory vector for the LGM odd basis, half the sum of the columns, is $(a \ a \ a \ a \ a-2)$.

In section 6.3, the method for translating the canonical LGM odd basis of an even-degree graph family into canonical LGM format was described. It involved subtracting the involutory vector from the first row vector, followed by further vector subtractions. The first subtraction has the effect of halving the determinant of the LGM. To obtain the Rogers' format LGM from the LGM odd basis, the method is simplified:

$$\begin{array}{c} \text{Rogers' format LGM} \\ \left(\begin{array}{ccccc} 2a & 0 & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 & 0 \\ 0 & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & 2a & 0 \\ a & a & a & a & a-2 \end{array} \right) \end{array}.$$

The involutory vector is subtracted from the last vector instead of the first, and then the signs of the last row and last column are reversed. Sharing a common LGM odd basis ensures that the two lattices are identical and hence that the corresponding even-degree graph families are isomorphic.

Another possible source of Abelian Cayley graph families with order higher than quasimaximal are lattices associated with optimal packings or coverings in Euclidean hyperspace. In particular, there are lattices for optimal sphere packings that are *tight* in the sense that each sphere is surrounded by the maximum possible number of touching spheres packed so tightly that there is no room for any of them to move. Such a case is only known for four dimensions: 1, 2, 8 and 24. Dimension 1 is trivial. Dimension 2 is a hexagonal tiling of circles in the plane. Dimension 8 are hyperspheres centred on the points of the E8 (or Gosset) lattice, where each hypersphere is touched by 240 others. Dimension 24 is the famous Leech lattice, where the hypersphere on each lattice point is touched by 196,560 others. These established properties are covered by Conway and Sloane [5].

Optimal sphere packing does not imply an optimal sphere covering. Also hyperspheres in Euclidean geometry do not translate directly to integer hyperspace and Lee spheres with the Manhattan norm. Nevertheless, it was considered worthwhile to explore these two lattices, which are both lower-triangular and have a similar format to Rogers'.

The E8 lattice, in its simplest integer form, has determinant value $2^8 = 256$ and is amenable to computer search. It is converted into an LGM by multiplying each element by the parameter a :

$$\begin{pmatrix} 4a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2a & 2a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2a & 2a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2a & 2a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2a & 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2a & 2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2a & 2a & 0 \\ a & a & a & a & a & a & a & a \end{pmatrix}.$$

In 8-dimensional Euclidean space, each unit cell of the E8 lattice contains two types of holes: 16 deep holes and 9 shallow holes. The deep holes have a depth of $2a$ and an example is $(2a \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$. The shallow holes, such as $(5a/3 \ a/3 \ a/3 \ a/3 \ a/3 \ a/3 \ a/3 \ a/3)$, have a depth of $4\sqrt{2}a/3$. Two of the lattice points at this distance from this shallow hole are $(0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$ and $(a \ a \ a \ a \ a \ a \ a \ a)$. However, with the Manhattan norm, the holes are altered. The deep hole above retains its depth of $2a$ under the Manhattan norm. But the distance from the shallow hole to the two lattice points is no longer $4\sqrt{2}a/3$; instead the respective Manhattan distances are $4a$ and $16a/3$. So the Euclidean deep hole has become a Manhattan shallow hole, and the Euclidean shallow hole is now deeper, with the Manhattan norm, than the deep hole and is no longer actually a hole. The Euclidean shallow hole has integer coordinates only when a is a multiple of 3. So in general, the holes are further altered when considering not only Manhattan norm but also integer rather than real hyperspace.

We now consider the E8 LGM under alternative assumptions of space and distance, and implications for the parameters of a hypothetical corresponding Abelian Cayley graph family. The E8 LGM has determinant $256a^8$, representing the order n of the corresponding graph family. In Euclidean space, the lattice deep hole depth is $2a$, taken to represent the corresponding diameter k . Then we have $n = k^8$. This would be a significantly better result than the Extremal Order Conjecture with leading term

$(1/2)(4/f)^f k^f$ for $f = 8$: $(1/2^9)k^8 \approx 1.95 \times 10^{-3}k^8$. If, instead, we use the Manhattan norm and diameter $k = 4a$, then we have $n = (1/2^8)k^8$, equal to twice the leading term of the Extremal Order Conjecture. However, moving from real space to integer space, it emerges that there are more deep holes and they are deeper. For values of a from 1 to 5, the lattice was searched to determine the number of deep holes in each unit cell (using the Manhattan norm) and the depth of these holes. The results are presented in Table 13.9.

Table 13.9: Deep holes in the E8 lattice (Manhattan norm)

Parameter a	Determinant n	Number of deep holes	Depth of deep holes (Manhattan norm)
1	256	70	4
2	65536	64	10
3	1679616	4480	14
4	16777216	64	20
5	100000000	4480	24

In Euclidean space, the scale of the lattice does not affect the number of deep holes or the form of the covering, and the unimodular E8 lattice has 16 deep holes per unit cell with a depth of 1. However, in integer space with the Manhattan norm there are two distinct cases, depending on the parity of a . For even a , there are 64 deep holes with Manhattan depth $5a$. For odd a , there are 4480 deep holes with depth $5a - 1$, except that many of them are coincident for $a = 1$. This translates into two families of Abelian Cayley graphs, with diameter class defined by deep-hole depth.

Unfortunately, although the leading coefficient of their order polynomial in $2a$ is 1, compared with $1/2$ for quasimaximal graph families, these families are subquasimaximal as their diameter $k = 5a + c$ for appropriate constant c , compared with $k = 4a + c$ for quasimaximal families, see Table 13.10. These Abelian Cayley graph families have leading term $(2/5)^8 k^8 \approx 6.55 \times 10^{-4} k^8$, about one third of the Extremal Order Conjecture value.

Table 13.10: E8 lattice graph families compared with largest-known Abelian Cayley graph families

Lattice	Diameter k	Parameter a	Order polynomial in $2a$	Comparable orders for $k = 10$ and 14
E8	0 (mod 10)	$k/5$	(1 0 0 0 0 0 0 0)	65536 ($k = 10$)
E8	4 (mod 10)	$(k + 1)/5$	(1 0 0 0 0 0 0 0)	1679616 ($k = 14$)
Largest -known	0 (mod 4)	$k/4$	(1 2 20 28 11 2 -4 -4 0)/2	
Largest -known	2 (mod 4)	$(k + 2)/4$	(1 -6 33 -100 183 -212 151 -60 10)/2	479255 ($k = 10$) 5109237 ($k = 14$)

The 24-dimensional Leech lattice is made unimodular by dividing each element of its simplest integer representation by $\sqrt{8}$. The determinant of the smallest integer version is a rather unwieldy $2^{36} = 68,719,476,736$, and the second over 4×10^{21} , which is too large for investigation by computer. Multiplying each element by a gives an LGM with determinant $n = 2^{36}a^{24}$. In Euclidean space, the lattice deep-hole depth is $k = 4a$, see [4], giving $n = (1/2)^{12}k^{24} \approx 2.44 \times 10^{-4}k^{24}$. In comparison, the Extremal Order Conjecture leading term, $(1/2)(4/f)^fk^f$, gives $(1/2)(1/6)^{24}k^{24} \approx 1.05 \times 10^{-19}k^{24}$. The deep-hole depth by the Manhattan norm depends on how many of the dimensions are involved in the shortest path from a lattice point. For example, a path of length a in each of 12 dimensions and $2a/3$ in 9 dimensions would also have a Euclidean length of $4a$ but a Manhattan length of $18a$. Taking $k = 18a$ gives $n = (2^{36}/18^{24})k^{24} \approx 5.13 \times 10^{-20}k^{24}$, about half the Extremal Order Conjecture leading term. Unfortunately, because of the large size of its determinant, it has not been possible to search for its Manhattan deep holes in integer space. Instead, the above value of $18a$ is taken as a proxy for their depth.

With both the E8 and the Leech lattices, the LGMs constructed from their matrices indicate corresponding Abelian Cayley graphs that are broadly in line with the Extremal Order Conjecture. The E8 family has been shown to be subquasimaximal, and the Leech family is likely to be.

A final observation about Rogers' result, casting some doubt on its validity, is an inconsistency in definition of the hypercube edge length between some supporting theorems used in the proof of Theorem 5.8 in his 1964 book [47]:

- In Theorem 5.5, the edge length of the 1-dimensional hypercube, C , is defined to be 2.
- The proof of Theorem 5.6 uses Theorem 5.5 to step inductively from dimension 1 to h , but assumes a hypercube edge length of 1 instead of 2.
- The proof of Theorem 5.8 uses Theorem 5.6, but with hypercube edge length that varies depending on the convex polytope of the lattice covering.

In summary, four aspects cast doubt on the validity of Rogers result:

- The key to Rogers' approach is the definition of hypercylinders, formed with hypercubes, inscribed within the convex polytope. However, these hypercubes are defined only for a vanishingly small proportion of the dimensions. The construction for the majority of the dimensions is not defined.
- Rogers' proof depends on a set of theorems with inconsistent assumptions about the edge lengths of the hypercubes.

- Rogers' lattice generator matrices have an essentially canonical quasimaximal format. For the low dimensions investigated, they generate quasimaximal Abelian Cayley graph families with order polynomials that are not extremal but have first two coefficients that are consistent with the Extremal Order Conjecture 3.1.
- Rogers' proof is not constructive. During more than 60 years since Rogers' paper [46], no construction of such a lattice covering has been published in the literature.

Gritzmann's [19] and Ordentlich, Regev and Weiss's [43] results depend on one of Roger's theorems. Therefore, it is considered that the Extremal Order Conjecture for Abelian Cayley graphs, Conjecture 3.1, remains valid.

GRAPH FAMILIES OF DIAMETER 2 AND ABOVE OF ARBITRARY DEGREE

In contrast to all the other chapters, in Chapter 14, graph families are considered where the diameter is fixed instead of the degree. Graph families with diameter 2 and arbitrary degree are discussed. Some improved lower and upper bounds are established for their extremal order.

14.1 Extremal diameter 2 circulant graph orders

We have seen that for every given degree investigated, the extremal and largest-known circulant graphs for each diameter above a low threshold have order determined by a periodic sequence of polynomials defined by a set of graph families. These polynomials have degree equal to the dimension of the graphs.

It is interesting to consider the conjugate problem of the sequence of orders of the same graphs as the degree increases for any given diameter. Algebraic constructions that aim to generate sequences of largest-known circulant graphs have all taken this approach, so far only for diameters 2 and 3. The ones based on direct products of Galois fields are only valid for a sparse sets of degrees, and none of them achieves extremal orders. We will consider the main diameter 2 constructions from the literature and their asymptotic limits in Section 14.3.

For diameter 2, the extremal circulant graphs for each degree do not have order determined by a periodic sequence of polynomials defined by a set of graph families. Instead the sequence of extremal orders appears chaotic or random. Our more limited investigation of graphs of increasing degree for given fixed diameter 3 and above has found a similar apparent chaotic behaviour. If there is any underlying pattern or structure to these sequences, then this emerges at higher degrees than has been studied to date. However, it is conjectured that there is no such structure and that the sequence of orders of extremal circulant graphs of increasing degree for any fixed diameter is chaotic throughout, and that this applies also to Abelian Cayley graphs.

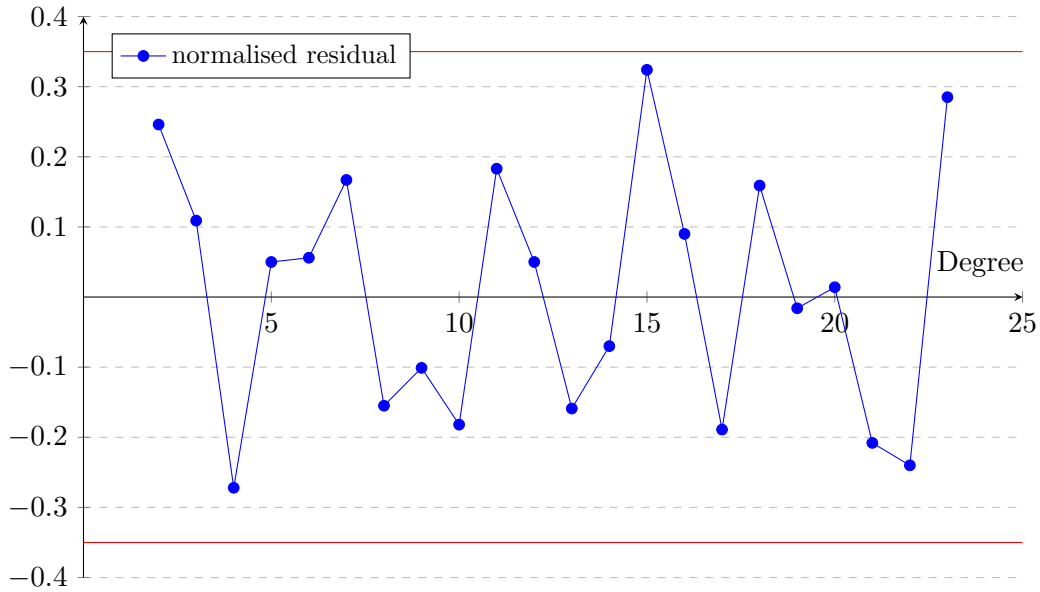
Conjecture 14.1. *For any given diameter $k \geq 2$, there is no finite set of degree classes $\{D_1, \dots, D_g\}$ where for any degree d , $d \in D_i$ for some i , and no set of*

circulant graph families $\{F_1, \dots, F_g\}$ defined for each degree class and with order $n_1(d), \dots, n_g(d)$ respectively, where $n_i(d)$ is a polynomial of degree k , such that for any i and any $d \in D_i$, $Ext_{circ}(d, k) = n_i(d)$. Similarly for extremal Abelian Cayley graphs of diameter $k \geq 2$.

Notwithstanding this chaotic behaviour, each sequence studied displays limited chaotic variation about a single polynomial of degree equal to the diameter. We will demonstrate this for the case of diameter 2, for which extremal circulant graphs up to degree 23 have been discovered (see Appendix D).

The best least-squares fit of a quadratic polynomial in the degree to the orders of the extremal diameter 2 circulant graphs up to degree 23 has a quadratic coefficient of 0.375 to three significant figures, which is $3/8$. This polynomial, to three significant figures, is $\hat{n}(d) = 0.375d^2 + 0.961d + 2.07$. A graph of the residual (divided by the degree to normalise), $(\hat{n}(d) - Ext_{circ}(d, 2))/d$, is shown in Figure 14.1.

Figure 14.1: Diameter 2 circulant graph order, up to degree 23: least-squares residual divided by the degree



The absolute value of this residual term remains below 0.35, as indicated by the parallel red lines. If this were to remain true for every higher degree, then this would imply the following lower and upper bounds on $Ext_{circ}(d, 2)$:

$0.375d^2 + 0.611d + 2.07 < Ext_{circ}(d, 2) < 0.375d^2 + 1.311d + 2.07$, and consequently that the leading coefficient is precisely $3/8$, giving

$$Ext_{circ}(d, 2) = \frac{3}{8}d^2 + O(d).$$

Extending the analysis to general Abelian Cayley graphs of diameter 2 does not significantly change the quadratic coefficient of the fitted polynomial. We have extremal orders, either verified or conjectured, for Abelian Cayley graphs only up to degree 20 (see Appendix F). Across this range, the resultant coefficient is 0.373 to three significant figures - also equal to $3/8$ within the margin of error. All the extremal non-circulant Abelian Cayley graphs up to degree 20 happen to have cyclic rank 2. This is a consequence of the restrictions on valid cyclic orders for graphs of higher cyclic rank with such low orders. We might expect graphs of higher cyclic rank to become increasingly dominant as the degree increases. The orders of the extremal and largest-known diameter 2 Abelian Cayley graphs up to cyclic rank 4 and degree 20, as far as discovered, are shown in Table 14.1.

Table 14.1: Order and cyclic order of extremal and largest-known diameter 2 Abelian Cayley graphs by degree and cyclic rank

Degree	Circulant Cyclic rank 1	Non-circulant Cyclic rank 2	Cyclic rank 3	Cyclic rank 4
2	5	-	-	-
3	8	-	-	-
4	13	$8 = 4 \times 2$	-	-
5	16	$16 = 4 \times 4$	-	-
6	21	$18 = 6 \times 3$	-	-
7	26	$24 = 12 \times 2$	$16 = 4 \times 2 \times 2$	-
8	35	$36 = 12 \times 3$	$16 = 4 \times 2 \times 2$	-
9	42	$40 = 20 \times 2$	$32 = 8 \times 2 \times 2$	-
10	51	$49 = 7 \times 7$	$48 = 12 \times 2 \times 2$	-
11	56	$56 = 28 \times 2$	$48 = 12 \times 2 \times 2$	$32 = 4 \times 2 \times 2 \times 2$
12	67	$72 = 12 \times 6$ & 36×2	$64 = 16 \times 2 \times 2$	$48 = 6 \times 2 \times 2 \times 2$
13	80	$80 = 40 \times 2$	$64 = 16 \times 2 \times 2$ & $4 \times 4 \times 4$	$64 = 8 \times 2 \times 2 \times 2$ & $4 \times 4 \times 2 \times 2$
14	90	$90 = 30 \times 3$	$80 = 20 \times 2 \times 2$	$64 = 8 \times 2 \times 2 \times 2$ & $4 \times 4 \times 2 \times 2$
15	96	$100 = 20 \times 5$	$80 = 20 \times 2 \times 2$	$64 = 8 \times 2 \times 2 \times 2$ & $4 \times 4 \times 2 \times 2$
16	112	$108 = 54 \times 2$ & 36×3	$108 = 12 \times 3 \times 3$ & $6 \times 6 \times 3$?
17	130	$120 = 60 \times 2$	$120 = 30 \times 2 \times 2$?
18	138	$147 = 21 \times 7$	$135 = 15 \times 3 \times 3$?
19	156	$156 = 78 \times 2$?	?
20	171	$168 = 84 \times 2$?	?

Notes: 1) Extremal graphs in bold
2) - does not exist
3) ? has not been investigated

14.2 Improved diameter 2 lower bounds for all degrees

By a simple counting argument, the upper bound for Abelian Cayley graphs of diameter 2 and degree d is seen to be

$$Upp_{AbCay}(d, 2) = \begin{cases} (d^2 + 2d + 2)/2 & \text{for } d \equiv 0 \pmod{2} \\ (d^2 + 2d + 1)/2 & \text{for } d \equiv 1 \pmod{2}. \end{cases}$$

The trivial lower bound has quadratic coefficient $1/4$. Various authors have published general constructions (valid for all d), but none has a quadratic coefficient larger than $1/4$. We briefly review them here and present a new largest-known general construction.

Constructions for Abelian Cayley graphs of diameter 2 and arbitrary degree d were given by Griggs in 1996 [18]. They have order $n = (d^2 + 4d + \delta)/4$, but are not circulant.

Dougherty and Faber derived circulant variants of these constructions in 2004 [10], with order differing by at most a constant. In the same paper, they presented two improved constructions with higher linear coefficients. Construction A with order $(d^2 + 6d + \delta)/4$, and Construction B with order $(d^2 + 8d + \delta)/4$. For construction B, the order n is given by

$$n = \begin{cases} (d^2 + 8d - 4)/4, & \text{for } d \equiv 0 \pmod{4} \\ (d^2 + 8d)/4, & \text{for } d \equiv 2 \pmod{4} \\ (d^2 + 8d - 1)/4, & \text{for } d \equiv 1 \pmod{2}. \end{cases}$$

In 2013, Monakhova published new constructions for even degree that improve on Dougherty and Faber's for $d \equiv 0 \pmod{4}$ [39]. The order n is given by

$$n = \begin{cases} (d^2 + 8d + 4)/4, & \text{for } d \equiv 0 \pmod{4} \\ (d^2 + 8d)/4, & \text{for } d \equiv 2 \pmod{4}. \end{cases}$$

We present constructions for four new families with linear coefficient increased from 2 to 3. The existence of these families has been verified by computer search for all degrees up to 30,000. They are conjectured to exist for all higher degrees.

Construction 14.2. *For every degree $d \leq 30000$, there exists a circulant graph of diameter 2 and order n given by*

$$n = \begin{cases} (d^2 + 12d - 28)/4, & \text{for } d \equiv 0 \pmod{4} \\ (d^2 + 12d - 32)/4, & \text{for } d \equiv 2 \pmod{4} \\ (d^2 + 12d - 45)/4, & \text{for } d \equiv 1 \pmod{2}. \end{cases}$$

The generating set for each degree class $d \pmod{4}$ is made up of

$$\begin{aligned} (s+1)/2 \text{ elements: } & 1, 3, \dots, s \\ u \text{ elements: } & s+t, s+2t, \dots, s+ut \\ 1 \text{ element: } & s+(u+1)t-2 \end{aligned}$$

$$\text{where } \begin{cases} s = (d-2)/2, t = (d+6)/2, u = (d-4)/4 & \text{for } d \equiv 0 \pmod{4} \\ s = (d-3)/2, t = (d+5)/2, u = (d-5)/4 & \text{for } d \equiv 1 \pmod{4} \\ s = d/2, t = (d+8)/2, u = (d-6)/4 & \text{for } d \equiv 2 \pmod{4} \\ s = (d-1)/2, t = (d+7)/2, u = (d-7)/4 & \text{for } d \equiv 3 \pmod{4}. \end{cases}$$

These graphs establish an improved general lower bound (verified for any degree $10 \leq d \leq 30,000$). However, the quadratic coefficient remains equal to the trivial lower bound at $1/4$. In the next section, we consider solutions for sparse sets of valid degrees that improve on this coefficient.

14.3 Improved upper bounds for certain classes of graph families

In this section, we remove the requirement that solutions be valid for all degrees and consider circulant and Abelian Cayley graph families of diameter 2 for sparse but infinite sets of degrees. These results were published in a paper [25].

As the Abelian Cayley upper bound for diameter 2, and any degree d , we have $Upp_{AbCay}(d, 2) = \lfloor \frac{1}{2}d^2 + d + 1 \rfloor$. For a lower bound, $LK_{circ}(d, 2)$, we have the new circulant graph family with order $\frac{1}{4}d^2 + 3d + \delta$ in the range $10 \leq d \leq 30,000$. For all other degrees, we have order $\frac{1}{4}d^2 + 2d + \delta$ from Dougherty and Faber [10] and Monokhova [38]. Thus, we have

$$\frac{1}{4}d^2 + 2d + \delta \leq Ext_{circ}(d, 2) \leq Ext_{AbCay}(d, 2) \leq \left\lfloor \frac{1}{2}d^2 + d + 1 \right\rfloor \text{ for any } d.$$

The gap between the quadratic coefficients of the lower bound, $1/4$, and the upper bound, $1/2$, is disappointingly large. In 2012 Macbeth, Šiagiová and Širáň established

a better lower bound for circulant graphs valid for a sparse but infinite set of degrees [32]. Their family of solutions has order $n = 9(d^2 + d - 6)/25$ for degree $d = 5p - 3$, where p is a prime with $p \equiv 2 \pmod{3}$, giving $Ext_{circ}(d, 2) \geq (9/25)d^2 + O(d)$, with a quadratic coefficient of 0.360, for admissible values of d . This was achieved by constructing the direct product of three cyclic groups of pairwise coprime order $F^* \times F^+ \times \mathbb{Z}_9$, where $F = GF(p)$ is the Galois field of order p , with additive group F^+ and multiplicative group F^* , and selecting an appropriate generating set for the Cayley graph.

Vetrík extended this method to establish a slightly improved lower bound for a different infinite set of degrees [50]. The graphs have order $n = 13(d^2 - 2d - 8)/36$ for degree $d = 6p - 2$, where p is a prime, $p \neq 13$, and $p \not\equiv 1 \pmod{13}$, giving $Ext_{circ}(d, 2) \geq (13/36)d^2 + O(d)$, with a quadratic coefficient of about 0.361, for admissible values of d . The construction also involves the direct product of three cyclic groups of coprime order, $F^* \times F^+ \times \mathbb{Z}_{13}$.

By relaxing the specification of the type of graph from circulant to any Abelian Cayley, an improved lower bound giving $Ext_{AbCay}(d, 2) \geq (3/8)d^2 + O(d)$, with a quadratic coefficient of 0.375 was identified by Macbeth, Šiagiová and Širáň [32]. This again involves the direct product of three cyclic groups, but this time their orders are not coprime so that the group is Abelian but not cyclic. The construction uses $F^* \times F^+ \times \mathbb{Z}_6$, where $F = GF(p)$ is the Galois field of order an odd prime power p .

The question arises whether other constructions of a similar form $F^* \times F^+ \times \mathbb{Z}_q$, where $F = GF(p)$ is the Galois field of order p , and $q \in \mathbb{N}$, might provide improved lower bounds, with quadratic coefficient above 13/36 for an infinite set of diameter 2 circulant graphs or above 3/8 for Abelian Cayley graphs. In the following sections, we consider generalisations of this method for diameter 2 circulant and Abelian Cayley graphs.

14.3.1 Circulant graphs for groups of the form $G = F^* \times F^+ \times \mathbb{Z}_q$

We first consider circulant graphs and a generalisation of the approach taken by Macbeth, Šiagiová and Širáň and by Vetrík based on the cyclic group $G = F^* \times F^+ \times \mathbb{Z}_q$, where $F = GF(p)$ for prime p , and $q = 9$ and 13 respectively. For the generalisation, we will consider this group for any odd $q \in \mathbb{N}$. Beforehand, some important component sets of the connection set C are defined.

Definition 14.3. For any $q \in \mathbb{N}$, $x \in F^*$, $y \in F^+$, $u, v, w \in \mathbb{Z}_q$, let $a_u(x) = (x, x, u)$, $b_v(x) = (x, 0, v)$ and $c_w(y) = (1, y, w)$, with $a_u^{-1}(x) = (x^{-1}, -x, -u)$, $b_v^{-1}(x) = (x^{-1}, 0, -v)$, and $c_w^{-1}(y) = (1, -y, -w)$. For any $u, v, w \in \mathbb{Z}_q$ we also define

$A_u = \{a_u(x), a_u^{-1}(x) : x \in F^*\}$, $B_v = \{b_v(x), b_v^{-1}(x) : x \in F^*\}$, $C_w^* = \{c_w(y), c_w^{-1}(y) : y \in F^*\}$ and $C_w^+ = \{c_w(y), c_w^{-1}(y) : y \in F^+\}$.

Clearly, for $u, v, w \neq 0$, A_u , B_v and C_w^* have size $2(p-1)$ and C_w^+ has size $2p$. For $u, v, w = 0$, we see that A_0 has size $2(p-1)$, B_0 and C_0^* have size $p-1$ and C_0^+ has size p .

In Macbeth, Šiagiová and Širáň's construction with $q = 9$, the connection set C is comprised of the sets A_1, B_3 and C_0^* along with two other elements, and hence $|C| = 5p - 3$. It is relatively straightforward to prove that any element of G can be expressed as the sum of at most two elements of C so that the resultant Cayley graph $X = (G, C)$ has diameter 2. As the degree of the graph $d = |C|$, we have $p = (d+3)/5$. Thus, the Cayley graph order $|G| = 9p(p-1) = (9/25)(d+3)(d-2)$, giving the quadratic coefficient $9/25$. A necessary condition for the construction is that every element of \mathbb{Z}_9 can be realised as the sum or difference of two of the subscripts of A_1, B_3 and C_0^* , that is $\pm 1, \pm 3$ and 0 . It is also necessary that each of these pairs of sets generate all but at most a constant number of the elements of $F^* \times F^+$, with the exceptions covered separately. In Vetrík's construction with $q = 13$, the connection set C includes the sets A_1, B_3 and C_4^+ along with two other elements, thus $|C| = 6p - 2$. Again, we find that any element of G can be expressed as the sum of at most two elements of C so that the resultant Cayley graph $X = (G, C)$ has diameter 2. In this case, we have $p = (d+2)/6$. Thus, the order of the Cayley graph $|G| = 13p(p-1) = (13/36)(d+2)(d-4)$.

In both cases, there is a relation between the degree d and the prime number p of the form $d = lp + \delta$ for constants l, δ (with $l = 5$ and $l = 6$ respectively) generating a graph of order $|G| = (q/l^2)d^2 + O(d)$. For the generalisation of this approach, we take the component sets A_u, B_v, C_w^+ . For any $l \geq 3$, we consider different values of q and for each q , the corresponding family of cyclic groups $G_l = F^* \times F^+ \times \mathbb{Z}_q$ for any prime $p = |F|$ such that $p-1, p, q$ are pairwise coprime. For each G_l , we consider connection sets C comprised of all possible combinations of the sets A_u, B_v, C_w^+ , along with a fixed number of other elements of G_l such that $|C| = lp + \delta$ for some fixed δ , with the condition that the Cayley graph $X(G_l, C)$ has diameter 2. There is no value in including A_0 in C as any element $(x, y, 0)$ is the sum of two elements of A_u for any $u \neq 0$, as we shall see later. There is no material difference between including B_0 or C_0^+ and no value in including both. So we will consider C_0^+ as the only set with subscript 0 for possible inclusion in C . Thus, if l is odd C_0^+ is included in C and if l is even C_0^+ is not included. For any l , we define q_l to be the largest value of q for which such a Cayley graph $X(G, C)$ exists for all admissible p , and define

$G_l = F^* \times F^+ \times \mathbb{Z}_{q_l}$. It follows that the order of $X(G_l, C)$ is $(q_l/l^2)d^2 + O(d)$, and we denote this order by $n_l(d)$.

Definition 14.4. Let $Ext_{circ}^{Gal}(d, 2)$ be the largest order of a circulant graph of diameter 2 and degree d , constructed as the Cayley graph of a group of the form $G = F^* \times F^+ \times \mathbb{Z}_q$.

Then $Ext_{circ}^{Gal}(d, 2) = \sup_{l \geq 3} n_l(d)$. Theorem 14.5 establishes an improved upper bound for such graphs, with quadratic coefficient $3/8$.

Theorem 14.5. With $Ext_{circ}^{Gal}(d, 2)$ as defined above, $Ext_{circ}^{Gal}(d, 2) \leq (3/8)d^2 + O(d)$.

Proof. Let F^* be the multiplicative group and let F^+ be the additive group of the Galois field $GF(p)$, where p is a prime such that $(p, q) = 1$ and $(p - 1, q) = 1$, so that $p, p - 1$ and q are pairwise coprime. Let $G = F^* \times F^+ \times \mathbb{Z}_q$. Since F^* , F^+ and \mathbb{Z}_q are cyclic groups of coprime order, the group G is also cyclic. Let 0 denote the identity in F^+ and \mathbb{Z}_q , and 1 the identity in F^* .

Consider the Cayley graph, $X(G, C)$, of the group G with a connection set C that includes the union of A_u , B_v and C_w^+ for multiple non-zero values of u, v and w , numbering m in total. Consider A_u for $u \in U$, B_v for $v \in V$ and C_w^+ for $w \in W$ where U, V and W are index sets with $g_a = |U|$, $g_b = |V|$ and $g_c = |W|$, so that $g_a + g_b + g_c = m$. C is constructed to be inverse-closed so that $X(G, C)$ is a circulant graph of degree $d = |C|$. Later, we will also consider including the inverse-closed set C_0^+ , and we define $l = 2m$ if C_0^+ is not in C and $l = 2m + 1$ if C_0^+ is in C . We now investigate how elements of G may be constructed from pairs of the sets A_u, B_v, C_w^+ .

First, consider B_v and C_w^+ for $v \in V$ and $w \in W$. For any $x \in F^*$, $y \in F^+$ we have

$$\begin{aligned} (x, y, v + w) &= (x, 0, v)(1, y, w) &= b_v(x)c_w(y) \\ (x, y, v - w) &= (x, 0, v)(1, y, -w) &= b_v(x)c_w^{-1}(-y) \\ (x, y, -v + w) &= (x, 0, -v)(1, y, w) &= b_v^{-1}(x^{-1})c_w(y) \\ (x, y, -v - w) &= (x, 0, -v)(1, y, -w) &= b_v^{-1}(x^{-1})c_w^{-1}(-y). \end{aligned}$$

Next, consider A_u and C_w^+ for $u \in U$ and $w \in W$. For any $x \in F^*$, $y \in F^+$ we have

$$\begin{aligned} (x, y, u + w) &= (x, x, u)(1, y - x, w) &= a_u(x)c_w(y - x) \\ (x, y, u - w) &= (x, x, u)(1, y - x, -w) &= a_u(x)c_w^{-1}(x - y) \\ (x, y, -u + w) &= (x, -x^{-1}, -u)(1, y + x^{-1}, w) &= a_u^{-1}(x^{-1})c_w(y + x^{-1}) \\ (x, y, -u - w) &= (x, -x^{-1}, -u)(1, y + x^{-1}, -w) &= a_u^{-1}(x^{-1})c_w^{-1}(-y - x^{-1}). \end{aligned}$$

Now consider A_u and B_v for $u \in U$ and $v \in V$. For any $x \in F^*$, $y \in F^*$ we have

$$\begin{aligned} (x, y, u + v) &= (y, y, u)(xy^{-1}, 0, v) &= a_u(y)b_v(xy^{-1}) \\ (x, y, u - v) &= (y, y, u)(xy^{-1}, 0, -v) &= a_u(y)b_v^{-1}(x^{-1}y) \\ (x, y, -u + v) &= (-y^{-1}, y, -u)(-xy, 0, v) &= a_u^{-1}(-y)b_v(-xy) \\ (x, y, -u - v) &= (-y^{-1}, y, -u)(-xy, 0, -v) &= a_u^{-1}(-y)b_v^{-1}(-x^{-1}y^{-1}). \end{aligned}$$

In case $y = 0$, introducing also $b_u(1)$ and $b_u^{-1}(1)$ for $u \in U$,

$$\begin{aligned} (x, 0, u + v) &= (1, 0, u)(x, 0, v) &= b_u(1)b_v(x) \\ (x, 0, u - v) &= (1, 0, u)(x, 0, -v) &= b_u(1)b_v^{-1}(x^{-1}) \\ (x, 0, -u + v) &= (1, 0, -u)(x, 0, v) &= b_u^{-1}(1)b_v(x) \\ (x, 0, -u - v) &= (1, 0, -u)(x, 0, -v) &= b_u^{-1}(1)b_v^{-1}(x^{-1}). \end{aligned}$$

Finally, consider A_u and $A_{u'}$ for $u, u' \in U, u \neq u'$. For any $x \in F^* \setminus \{1\}$, $y \in F^*$ we have

$$\begin{aligned} (x, y, u - u') &= (xy/(x-1), xy/(x-1), u)((x-1)/y, -y/(x-1), -u') \\ &= a_u(xy/(x-1))a_{u'}^{-1}(y/(x-1)). \end{aligned}$$

In case $x = 1$, where w' is any fixed element of W , and introducing also $b_{u-u'-w'}(1)$,

$$(1, y, u - u') = (1, y, w')(1, 0, u - u' - w') = c_{w'}(y)b_{u-u'-w'}(1).$$

And in case $y = 0$, we consider $(x, 0, u - u') = b_{u-u'}(x)$. If $u - u' = v$ for some $v \in V$ then this is immediately covered by $b_v(x)$. Otherwise, if $u - u' = v + v'$ for some $v, v' \in V$ then

$$(x, 0, u - u') = (x, 0, v + v') = (x, 0, v)(1, 0, v') = b_v(x)b_{v'}(1),$$

and similarly for $u - u' = v - v'$ or $-v - v'$. For any remaining uncovered cases $b_{u-u'}(x)$, it would be necessary to introduce $b_u(x)$ for any $x \in F^*$, using the construction

$$(x, 0, u - u') = (x, 0, u)(1, 0, -u') = b_u(x)b_{u'}^{-1}(1).$$

From the above, we see that each pair $A_u B_v$, $A_u C_w^+$, $B_v C_w^+$ can generate all elements of G containing up to four values of \mathbb{Z}_q with a limited number of exceptions that are covered by the additionally introduced elements as shown above. Similarly, each pair $A_u A_{u'}$ generates all elements of G containing up to two values of \mathbb{Z}_q . On the other hand, the first coordinate of the product of an element of C_w^+ with an element of C_w^+ ,

is always 1, and the second coordinate of the product of an element of B_v with an element of $B_{v'}$ is always 0. So these combinations do not contribute to an efficient covering of G . Summing the elements listed in the cases above gives the following lower bound for the degree d of the corresponding circulant graph:

$$\begin{aligned} d = |C| &\geq 2g_a(p-1) + 2g_b(p-1) + 2g_cp + 2g_a + g_a(g_a-1) \\ &= 2m(p-1) + 2g_c + g_a + g_a^2, \text{ as } m = g_a + g_b + g_c. \end{aligned}$$

An upper bound for the value of q is obtained by assuming there is no duplication between the values of $s+t$, $s-t$, $-s+t$ and $-s-t$ across all $s, t \in \{u, v, w\}$ for the given combinations of A_u , B_v and C_w^+ , with the exception that $(x, y, 0)$ can be created from two elements of any A_u . Thus, the upper bound for q is given by

$$\begin{aligned} q &\leq 4g_ag_b + 4g_bg_c + 4g_ag_c + g_a(g_a-1) + 1 \\ &= 4g_am + 4g_bm - 3g_a^2 - 4g_b^2 - 4g_ag_b - g_a + 1, \text{ as } m = g_a + g_b + g_c. \end{aligned}$$

This is a maximum when the partial derivatives with respect to g_a and g_b are zero.

We have $\partial q/\partial g_a = 4m - 6g_a - 4g_b - 1 = 0$ and $\partial q/\partial g_b = 4m - 8g_b - 4g_a = 0$. Thus $g_a = (2m-1)/4$ and $g_b = g_c = (2m+1)/8$, and hence $q \leq (12m^2 - 4m + 9)/8$. Also we have $d \geq 2mp - 5m/4 + 1/16 + m^2/4$, so that $p \leq d/2m + 5/8 - 1/(32m) - m/8$. Thus, we find

$$\begin{aligned} |G| &\leq (p-1)p(12m^2 - 4m + 9)/8 \\ &= [3/8 - (4m-9)/(32m^2)]d^2 + O(d) \\ &\leq (3/8)d^2 + O(d), \text{ for } m > 2. \end{aligned}$$

The exceptional cases, where $m \leq 2$, are easily evaluated separately, see Table 14.2.

If instead C also includes the set C_0^+ , so that

$d = |C| \geq 2g_a(p-1) + 2g_b(p-1) + (2g_c+1)p + 2g_a + g_a(g_a-1)$, then we find that the upper bound for n is given by

$$q \leq 4g_ag_b + 4g_bg_c + 4g_ag_c + 2g_a + 2g_b + g_a(g_a-1) + 1.$$

In this case, partial differentiation gives a maximum value at $g_a = m/2$,

$g_b = (m+1)/4$ and $g_c = (m-1)/4$. Then $q \leq (6m^2 + 4m + 5)/4$. Also

$d \geq (2m+1)p - m - 1/2 + m^2/4$, so that $p \leq d/(2m+1) + 1/2 - m^2/(8m+2)$. In

this case $|G| \leq [(6m^2 + 4m + 5)/(4(2m+1)^2)]d^2 + O(d) \leq (3/8)d^2 + O(d)$ for $m > 3$.

Again, the exceptional cases, where $m \leq 3$, are evaluated separately. \square

In both cases, with and without C_0^+ as a subset of the connection set C , the optimum values of g_a , g_b and g_c determined by differentiation are often not integral and the resultant value of q is never integral. However, in practice it appears possible to find values such that q achieves the highest odd integer below the calculated value. As stated earlier, this upper bound assumes it is possible to find a set of values for the u , v and w such that none of the pairwise combinations are duplicates. This has been found to be possible in every case for $m \leq 3$ and also for the case $m = 5$ without C_0^+ , but for no other values checked up to $m = 8$. A summary of the best results is presented in Table 14.2.

Table 14.2: Upper bounds and extremal values for the order of the circulant graph for $l \leq 17$

Pairs m	Sets l	Upper bound for q		Quadratic coefficient		Extremal value, q	Quadratic coefficient
		Real	Integer	Fraction	Decimal		
1	3	3.75	3	1/3	0.333	3	0.333
2	4	6.125	5	5/16	0.313	5	0.313
2	5	9.25	9	9/25	0.360	9	0.360
3	6	13.125	13	13/36	0.361	13	0.361
3	7	17.75	17	17/49	0.347	17	0.347
4	8	23.125	23	23/64	0.359	21	0.328
4	9	29.25	29	29/81	0.358	27	0.333
5	10	36.125	35	35/100	0.350	35	0.350
5	11	43.75	43	43/121	0.355	41	0.339
6	12	52.125	51	51/144	0.354	49	0.340
6	13	61.25	61	61/169	0.361	57	0.337
7	14	71.125	71	71/196	0.362	65	0.332
7	15	81.75	81	81/225	0.360	75	0.333
8	16	93.125	93	93/256	0.363	87	0.340
8	17	105.25	105	105/289	0.363	97	0.336

Table 14.3: A solution for each extremal value of the order of the circulant graph for $l \leq 17$

Pairs m	Sets l	Extremal value, q	Values of u for A_u	Values of v for B_v	Values of w for C_w
2	5	9	1	3	0
3	6	13	1	3	4
3	7	17	1, 8	3	0
4	8	21	2, 9	3	1
4	9	27	5, 11	12, 13	0
5	10	35	13, 16	7, 8	9
5	11	41	7, 17	2, 13	0, 1
6	12	49	19, 22, 23	1, 14	16
6	13	57	10, 24	3, 18	0, 1, 2
7	14	65	5, 11, 20, 27	6, 7	30
7	15	75	11, 20, 33	3, 26	0, 1, 2
8	16	87	5, 20, 29	40, 41, 42	13, 39
8	17	97	4, 31, 39	23, 24, 25	0, 13, 26

In most cases, there are multiple solutions for the connection set, including with different values for g_a , g_b and g_c . Table 14.3 shows one solution for each case. For $l = 5$, this is the solution given by Macbeth, Šiagiová and Širáň [32]. For $l = 6$, this is the solution given by Vetrík [50]. The quadratic coefficient of Vetrík's solution, $13/36$, is likely to be the best possible with this form of construction. Although it is possible for the upper bound to be arbitrarily close to $3/8$ for a large enough value of l , we see from Table 14.2 that the proportion of duplicates appears to increase with l , reaching about 7% by $l = 17$.

14.3.2 Abelian Cayley graphs for groups of the form $H = F^* \times F^+ \times \mathbb{Z}_q$

The circulant graph construction in the previous section can be extended to Abelian Cayley graphs $X(H, C)$ where $H = F^* \times F^+ \times \mathbb{Z}_q$ by relaxing the requirement that q is coprime with p and $p - 1$, for p a prime power, so that the connection set can include the self-inverse set $B_{q/2}$, thus requiring that q be even. In Macbeth, Šiagiová and Širáň's construction with $q = 6$, the connection set C is comprised of the sets A_1, B_3 and C_0^* along with two other elements, noting that B_3 is $B_{q/2}$, and hence $|C| = 4p - 2$. It is relatively straightforward to prove that any element of H can be expressed as the sum of at most two elements of C so that the resultant Cayley graph $X = (H, C)$ has diameter 2. As the degree of the graph $d = |C|$, we have $p = (d + 2)/4$. Thus, the order of the Cayley graph $|H| = 6p(p - 1) = (6/16)(d + 2)(d - 2)$, giving the quadratic coefficient $3/8$.

For the generalisation of this approach, we again take the component sets A_u, B_v, C_w^+ , as defined in Section 2. For any $l \geq 4$, we consider different values of q , and for each q the corresponding family of Abelian groups $H_l = F^* \times F^+ \times \mathbb{Z}_q$ where $F = GF(p)$ for any prime power p , and $q \in \mathbb{N}$. For each H_l , we consider connection sets C comprised of all possible combinations of the sets A_u, B_v, C_w^+ , always including $B_{q/2}$, along with a fixed number of other elements of H_l such that $|C| = lp + \delta$ for some fixed δ , with the condition that the Cayley graph $X(H_l, C)$ has diameter 2. As before we also consider the potential inclusion of the self-inverse set C_0 in the connection set. Thus as opposed to the circulant graph case, C_0^+ is included in C if l is odd and not included if l is even.

As before, for any l we define q_l to be the largest value of q for which such a Cayley graph $X(H, C)$ exists for all admissible p , and define $H_l = F^* \times F^+ \times \mathbb{Z}_{q_l}$. It follows that the order of $X(H_l, C)$ is $(q_l/l^2)d^2 + O(d)$, and we denote this order by $m_l(d)$.

Definition 14.6. Let $Ext_{AbCay}^{Gal}(d, 2)$ be the largest order of an Abelian Cayley graph of diameter 2 and degree d , constructed as the Cayley graph of a group of the form $H = F^* \times F^+ \times \mathbb{Z}_q$.

Then $Ext_{AbCay}^{Gal}(d, 2) = \sup_{l \geq 3} m_l(d)$. Theorem 14.7 establishes an improved upper bound for such graphs, with quadratic coefficient $3/8$.

Theorem 14.7. With $Ext_{AbCay}^{Gal}(d, 2)$ as defined above,
 $Ext_{AbCay}^{Gal}(d, 2) \leq (3/8)d^2 + O(d)$.

Proof. Let F^* be the multiplicative group and let F^+ be the additive group of the Galois field $GF(p)$, where p is a prime power, and let q be even. Let $H = F^* \times F^+ \times \mathbb{Z}_q$. Since F^* , F^+ and \mathbb{Z}_q are cyclic groups, the group H is Abelian.

Consider the Cayley graph, $X(H, C)$, of the group H with a connection set C that includes the union of A_u , B_v and C_w^+ for multiple values of u , v and w , not equal to 0 or $q/2$, numbering m in total. Consider A_u for $u \in U$, B_v for $v \in V$ and C_w^+ for $w \in W$ where U, V and W are index sets with $g_a = |U|$, $g_b = |V|$ and $g_c = |W|$, so that $g_a + g_b + g_c = m$. The set C includes $B_{q/2}$ and is constructed to be inverse-closed so that $X(H, C)$ is an Abelian Cayley graph of degree $d = |C|$. We also consider including C_0^+ . We define $l = 2m + 1$ if C_0^+ is not in C and $l = 2m + 2$ if C_0^+ is in C . We now investigate how elements of H may be constructed from pairs of the sets A_u , B_v , C_w^+ .

First, consider B_v and C_w^+ for $v \in V$ and $w \in W$, where $v \neq q/2, w \neq 0$. For any $x \in F^*$, $y \in F^+$ we have

$$\begin{aligned} (x, y, v + w) &= (x, 0, v)(1, y, w) &= b_v(x)c_w(y) \\ (x, y, v - w) &= (x, 0, v)(1, y, -w) &= b_v(x)c_w^{-1}(-y) \\ (x, y, -v + w) &= (x, 0, -v)(1, y, w) &= b_v^{-1}(x^{-1})c_w(y) \\ (x, y, -v - w) &= (x, 0, -v)(1, y, -w) &= b_v^{-1}(x^{-1})c_w^{-1}(-y). \end{aligned}$$

In case $v = q/2, w \neq 0$

$$\begin{aligned} (x, y, q/2 + w) &= (x, 0, q/2)(1, y, w) &= b_{n/2}(x)c_w(y) \\ (x, y, q/2 - w) &= (x, 0, q/2)(1, y, w) &= b_{n/2}(x)c_w(y). \end{aligned}$$

In case $w = 0, v \neq q/2$

$$\begin{aligned} (x, y, v) &= (x, 0, v)(1, y, 0) &= b_v(x)c_0(y) \\ (x, y, -v) &= (x, 0, -v)(1, y, 0) &= b_v^{-1}(x^{-1})c_0(y). \end{aligned}$$

And in case $v = q/2, w = 0$

$$(x, y, q/2) = (x, 0, q/2)(1, y, 0) = b_{n/2}(x)c_0(y).$$

Next, consider A_u and C_w^+ for $u \in U$ and $w \in W$, where $w \neq 0$. For any $x \in F^*$, $y \in F^+$ we have

$$\begin{aligned} (x, y, u + w) &= (x, x, u)(1, y - x, w) &= a_u(x)c_w(y - x) \\ (x, y, u - w) &= (x, x, u)(1, y - x, -w) &= a_u(x)c_w^{-1}(x - y) \\ (x, y, -u + w) &= (x, -x^{-1}, -u)(1, y + x^{-1}, w) &= a_u^{-1}(x^{-1})c_w(y + x^{-1}) \\ (x, y, -u - w) &= (x, -x^{-1}, -u)(1, y + x^{-1}, -w) &= a_u^{-1}(x^{-1})c_w^{-1}(-y - x^{-1}). \end{aligned}$$

In case $w = 0$

$$\begin{aligned} (x, y, u) &= (x, x, u)(1, y - x, 0) &= a_u(x)c_0(y - x) \\ (x, y, -u) &= (x, -x^{-1}, -u)(1, y + x^{-1}, 0) &= a_u^{-1}(x^{-1})c_0(y + x^{-1}). \end{aligned}$$

Now consider A_u and B_v for $u \in U$ and $v \in V$ where $v \neq q/2$. For any $x \in F^*$, $y \in F^*$ we have

$$\begin{aligned} (x, y, u + v) &= (y, y, u)(xy^{-1}, 0, v) &= a_u(y)b_v(xy^{-1}) \\ (x, y, u - v) &= (y, y, u)(xy^{-1}, 0, -v) &= a_u(y)b_v^{-1}(x^{-1}y) \\ (x, y, -u + v) &= (-y^{-1}, y, -u)(-xy, 0, v) &= a_u^{-1}(-y)b_v(-xy) \\ (x, y, -u - v) &= (-y^{-1}, y, -u)(-xy, 0, -v) &= a_u^{-1}(-y)b_v^{-1}(-x^{-1}y^{-1}). \end{aligned}$$

In case $v = q/2$

$$\begin{aligned} (x, y, q/2 + u) &= (y, y, u)(xy^{-1}, 0, q/2) &= a_u(y)b_{n/2}(xy^{-1}) \\ (x, y, q/2 - u) &= (-y^{-1}, y, -u)(-xy, 0, q/2) &= a_u^{-1}(y^{-1})b_{n/2}(-xy). \end{aligned}$$

For $y = 0$, where $v \neq q/2$, introducing also $b_u(1)$ and $b_u^{-1}(1)$ for $u \in U$,

$$\begin{aligned} (x, 0, u + v) &= (1, 0, u)(x, 0, v) &= b_u(1)b_v(x) \\ (x, 0, u - v) &= (1, 0, u)(x, 0, -v) &= b_u(1)b_v^{-1}(x^{-1}) \\ (x, 0, -u + v) &= (1, 0, -u)(x, 0, v) &= b_u^{-1}(1)b_v(x) \\ (x, 0, -u - v) &= (1, 0, -u)(x, 0, -v) &= b_u^{-1}(1)b_v^{-1}(x^{-1}). \end{aligned}$$

In case $v = q/2$

$$\begin{aligned} (x, 0, q/2 + u) &= (1, 0, u)(x, 0, q/2) &= b_u(1)b_{n/2}(x) \\ (x, 0, q/2 - u) &= (1, 0, -u)(x, 0, q/2) &= b_u^{-1}(1)b_{n/2}(x). \end{aligned}$$

Finally, consider A_u and A'_u for $u, u' \in U, u \neq u'$. For any $x \in F^* \setminus \{1\}, y \in F^*$ we have

$$\begin{aligned}(x, y, u - u') &= (xy/(x-1), xy/(x-1), u)((x-1)/y, -y/(x-1), -u') \\ &= a_u(xy/(x-1))a_{u'}^{-1}(y/(x-1)).\end{aligned}$$

In case $x = 1$, where w' is any fixed element of W , and introducing also $b_{u-u'-w'}(1)$,

$$(1, y, u - u') = (1, y, w')(1, 0, u - u' - w') = c_{w'}(y)b_{u-u'-w'}(1).$$

And in case $y = 0$, we consider $(x, 0, u - u') = b_{u-u'}(x)$. If $u - u' = v$ for some $v \in V$ then this is immediately covered by $b_v(x)$. Otherwise, if $u - u' = v + v'$ for some $v, v' \in V$ then

$$(x, 0, u - u') = (x, 0, v + v') = (x, 0, v)(1, 0, v') = b_v(x)b_{v'}(1)$$

and similarly for $u - u' = v - v'$ or $-v - v'$. For any remaining uncovered cases $b_{u-u'}(x)$, it would be necessary to introduce $b_u(x)$ for any $x \in F^*$, using the construction

$$(x, 0, u - u') = (x, 0, u)(1, 0, -u') = b_u(x)b_{u'}^{-1}(1).$$

From the above, we see that each pair $A_u B_v, A_u C_w^+, B_v C_w^+$, for $v \neq q/2$ and $w \neq 0$, can generate all elements of H with up to four values of \mathbb{Z}_q with a limited number of exceptions that are covered as shown above. Similarly, each pair $A_u A_{u'}, A_u C_w^+, B_v C_w^+$, for $u \neq u', v = q/2$ or $w = 0$, creates up to two values. On the other hand, the first coordinate of the product of an element of C_w^+ with an element of $C_{w'}^+$ is always 1, and the second coordinate of the product of an element of B_v with an element of $B_{v'}$ is always 0. So these combinations do not contribute to an efficient covering of H . Summing the elements listed in the cases above gives the following lower bound for the degree d of the corresponding Abelian Cayley graph:

$$\begin{aligned}d = |C| &\geq 2g_a(p-1) + (2g_b+1)(p-1) + 2g_c p + 2g_a + g_a(g_a-1) \\ &= (2m+1)(p-1) + 2g_c + g_a + g_a^2, \text{ as } m = g_a + g_b + g_c.\end{aligned}$$

An upper bound for the value of n is obtained by assuming there is no duplication between the values of $s+t, s-t, -s+t$ and $-s-t$ across all $s, t \in \{u, v, w\}$ for the given combinations of A_u, B_v and C_w^+ , with the exception that $(x, y, 0)$ can be created from two elements of any A_u . Thus, the upper bound for n is given by

$$\begin{aligned}q &= 4g_a g_b + 4g_b g_c + 4g_a g_c + g_a(g_a-1) + 2g_a + 2g_c + 1 \\ &= 4g_a m + 4g_b m - 3g_a^2 - 4g_b^2 - 4g_a g_b + m - g_a - 2g_b + 1, \text{ as } m = g_a + g_b + g_c.\end{aligned}$$

This is a maximum when the partial derivatives with respect to g_a and g_b are zero.

We have $\partial q/\partial g_a = 4m - 6g_a - 4g_b - 1 = 0$ and $\partial q/\partial g_b = 4m - 8g_b - 4g_a - 2 = 0$.

Thus, $g_a = m/2$, $g_b = (m-1)/4$ and $g_c = (m+1)/4$, and hence

$q \leq (6m^2 + 4m + 3)/4$. Also, we have $d \geq (2m+1)p + m^2/4 - m - 1/2$, so that

$p \leq d/(2m+1) - (m^2 - 4m - 2)/(8m+4)$. Thus, we find

$$\begin{aligned} |H| &\leq (p-1)p(6m^2 + 4m + 3)/4 \\ &= [(6m^2 + 4m + 3)/(4(2m+1)^2)]d^2 + O(d) \\ &\leq (3/8)d^2 + O(d), \text{ for } m \geq 1. \end{aligned}$$

If instead C also includes the set C_0^+ , so that

$d = |C| \geq 2g_a(p-1) + (2g_b+1)(p-1) + (2g_c+1)p + 2g_a + g_a(g_a-1)$, then we find that the upper bound for n is given by

$$\begin{aligned} q &\leq 4g_ag_b + 4g_bg_c + 4g_ag_c + 4g_a + 2g_b + 2g_c + g_a(g_a-1) + 2 \\ &= 4g_am + 4g_bm - 3g_a^2 - 4g_b^2 - 4g_ag_b + 2m + g_a + 2. \end{aligned}$$

In this case, partial differentiation gives a maximum value at $g_a = (2m+1)/4$,

$g_b = g_c = (2m-1)/8$. Then $q = (12m^2 + 20m + 17)/8$. Also

$d = 2(m+1)p + m^2/4 - 3m/4 - 15/16$, so that

$p = d/2(m+1) - (4m^2 - 12m - 15)/32(m+1)$, and we have $|H| \leq (3/8)d^2 + O(d)$ for $m \geq 1$. \square

As for the circulant graph case, the optimum values of g_a , g_b and g_c determined by differentiation are often not integral and the resultant value of q is never integral. The upper bound is then the largest even number below the calculated value. This is only achievable if it is possible to find a set of values for the u , v and w such that none of the pairwise combinations are duplicates. This has been found to be possible in every case for $m \leq 3$, as for the circulant graphs, but for no higher values checked up to $m = 7$. A summary of the best results is presented in Table 14.4.

It is interesting to note that for $l = 6$ we have extremal value $q = 12$ with a quadratic coefficient of $1/3$ (Table 14.4), whereas the corresponding circulant graphs have a higher extremal value, $q = 13$ with coefficient $13/36$ (Table 14.2). This is because the Abelian Cayley graph connection set is defined to include the two self-inverse sets $B_{q/2}$ and C_0^+ along with two pairs of non self-inverse sets, whereas the circulant graph connection set is comprised of three such pairs and neither of the self-inverse sets.

Without the requirement to include the set $B_{q/2}$, the extremal Abelian Cayley graph with this construction would be the circulant graph.

Table 14.4: Upper bounds and extremal values for the order of the Abelian Cayley graph for $l \leq 16$

Pairs m	Sets l	Upper bound for q		Quadratic coefficient		Extremal value, q	Quadratic coefficient
		Real	Integer	Fraction	Decimal		
1	4	6.125	6	3/8	0.375	6	0.375
2	5	8.75	8	8/25	0.320	8	0.320
2	6	13.125	12	1/3	0.333	12	0.333
3	7	17.25	16	16/49	0.327	16	0.327
3	8	23.125	22	11/32	0.344	22	0.344
4	9	28.75	28	28/81	0.346	26	0.321
4	10	36.125	36	9/25	0.360	34	0.340
5	11	43.25	42	42/121	0.347	40	0.331
5	12	52.125	52	13/36	0.361	48	0.333
6	13	60.75	60	60/169	0.355	56	0.331
6	14	71.125	70	35/98	0.357	66	0.337
7	15	81.25	80	16/45	0.356	72	0.320
7	16	93.125	92	23/64	0.359	86	0.336

Table 14.5: A solution for each extremal value of the order of the Abelian Cayley graph for $l \leq 16$

Pairs m	Sets l	Extremal value, q	Values of u for A_u	Values of v for B_v	Values of w for C_w
1	4	6	1	3	0
2	5	8	1	4	3
2	6	12	1	6	0, 3
3	7	16	1, 6	8	2
3	8	22	2, 7	11	0, 1
4	9	26	2, 9	13	1, 4
4	10	34	1, 8	2, 17	0, 13
5	11	40	12, 17, 18	1, 20	8
5	12	48	10, 18	3, 24	0, 1, 2
6	13	56	3, 4, 11	1, 26, 28	17
6	14	66	12, 26	3, 18, 33	0, 1, 2
7	15	72	3, 4, 11, 19	1, 34, 36	25
7	16	86	11, 26, 36	3, 20, 43	0, 1, 2

In most cases, there are multiple solutions for the connection set, including with different values for g_a , g_b and g_c . Table 14.5 shows one solution for each case. For $l = 4$, this is the solution given by Macbeth, Šiagiová and Širáň [32]. The quadratic coefficient of this solution, $3/8$, is likely to be unmatched for any other value of l with this form of construction. Although the upper bound is $3/8$, we see from Table 14.4 that the proportion of duplicates appears to increase with l , reaching about 7% by $l = 16$.

14.3.3 Extending the validity to any degree above a threshold

We note that the graphs from both constructions by Macbeth, Šiagiová and Širáň and from Vetrík's are only established for values of the degree that are a linear function of

a sequence of primes belonging to a prescribed congruence class or set of classes. For example, for Macbeth, Šiagiová and Širáň's circulant graph construction, the degree $d = 5p - 3$ where the prime $p \equiv 2 \pmod{3}$. A recent paper by Cullinan and Hajir [6] defines a method for identifying a bound on the length of interval that will always contain at least one prime of a prescribed congruence class. This builds on an earlier paper by Ramaré and Rumely [45] which includes a table, Table 1, that defines triples (k, x_0, ϵ) where k is the modulo of the congruence class, x_0 is a threshold minimum and ϵ is a corresponding factor. For any such triple, Cullinan and Hajir established that for any $x > x_0$ and $\delta > 2\epsilon/(1 - \epsilon)$ the interval $(x, x(1 + \delta)]$ will contain at least one prime $p \equiv a \pmod{k}$ for any a coprime with k . Therefore, in each of the graph constructions discussed, for any sufficiently large degree d it is possible to find a prime of the correct congruence class such that the corresponding degree $d' < d(1 + \delta)$. Within the table, x_0 takes four values, from 10^{10} to 10^{100} , with the lowest value of ϵ corresponding to $x_0 = 10^{100}$. In Theorem 14.8 below we take the largest value for x_0 in order to define the largest possible lower bound on the asymptotic value of the quadratic coefficient for each construction that is valid for any degree above the corresponding threshold.

Theorem 14.8. *Let $Ext_{circ}^{Gal}(d, 2)$ be defined as for Theorem 14.5 and $Ext_{AbCay}^{Gal}(d, 2)$ as for Theorem 14.7. Then we have the following*

$$\begin{aligned} 0.3581 &< \frac{Ext_{circ}^{Gal}(d, 2)}{d^2} \leq 0.3750 \text{ for any } d > 5 \times 10^{100}, \\ 0.3582 &< \frac{Ext_{circ}^{Gal}(d, 2)}{d^2} \leq 0.3750 \text{ for any } d > 6 \times 10^{100}, \\ 0.3749 &< \frac{Ext_{AbCay}^{Gal}(d, 2)}{d^2} \leq 0.3750 \text{ for any } d > 4 \times 10^{100}. \end{aligned}$$

Proof. Note that in this proof k does not denote the diameter, which is 2, but instead denotes the modulo of the congruence class referenced in Table 1 of Ramaré and Rumely's paper [45]. For the Macbeth, Šiagiová and Širáň circulant graph construction with quadratic coefficient $9/25$, we take the triple $k = 3, x_0 = 10^{100}, \epsilon = 0.001310$ from Table 1 of Ramaré and Rumely's paper, giving a value of $\delta = 0.002623$. Then $9/25 \times 1/(1 + \delta)^2 \approx 0.35811$. This is valid for $p > 10^{100}$, and hence for $d > 5 \times 10^{100}$ as $d < d' = 5p - 3$. For the Vetrík circulant graph construction with quadratic coefficient $13/36$, we take the triple $k = 13, x_0 = 10^{100}, \epsilon = 0.002020$ from the table, giving a value of $\delta = 0.004048$. Then $13/36 \times 1/(1 + \delta)^2 \approx 0.35821$, and we note that $d' = 6p - 2$. For the Macbeth, Šiagiová and Širáň Abelian Cayley graph construction with quadratic coefficient $3/8$,

we take the triple $k = 1, x_0 = 10^{100}, \epsilon = 0.000001$ from the table, giving a value of $\delta = 0.000002$. Then $3/8 \times 1/(1 + \delta)^2 \approx 0.37499$, and we note that $d' = 4p - 2$. \square

14.3.4 Upper bound quadratic coefficient

For circulant graphs of diameter 2, employing the method of construction with the direct product of the additive and multiplicative groups of a Galois field and cyclic group of any order, it proves to be impossible to achieve a quadratic coefficient for the order polynomial that is higher than $3/8$, thus improving the upper bound for this construction method from $1/2$ to $3/8$. For Abelian Cayley graphs constructed in the same way, the upper bound on the quadratic coefficient is also improved to a value of $3/8$. Applying Cullinan and Hajir's property of intervals containing a prime of a prescribed congruence class, the asymptotic value of the quadratic coefficient valid for every degree above a threshold exceeds 0.358 for the two circulant graph construction and remains at 0.375 to three significant figures for the Abelian Cayley graph construction.

All of the preceding constructions for diameter 2 circulant graphs and Abelian Cayley graphs have failed to exceed the value $3/8$ for the leading coefficient of the order quadratic in the degree. However, a recent paper by Pott and Zhou [44] presents two constructions using generalised difference sets with leading coefficient above $3/8$. They show that $\text{Ext}_{\text{AbCay}}(d, 2) \geq (25/64)d^2 - 2.1d^{1.525}$ for sufficiently large degree d , and also $\text{Ext}_{\text{AbCay}}(d, 2) \geq (4/9)d^2$ if $d = 3q, q = 2^m$ and m is odd. Prior to this achievement, the circumstantial evidence supported a conjecture that $3/8$ was an upper bound for diameter 2 Abelian Cayley graphs. With this limit being broken, the question arises whether the Moore bound coefficient of $1/2$ might be achievable with the right constructions. A related question is whether the $3/8$ limit can be exceeded also for circulant graphs.

14.4 Extremal graphs of higher fixed diameter

For extremal circulant graphs of diameter 2 and arbitrary degree d , we have seen three pieces of evidence indicating that a quadratic in d with leading coefficient of $3/8$ might be a good approximation to their order. Firstly, in Section 14.1, the best fit to the order of the known extremal circulant graphs is given by a quadratic with leading coefficient $3/8$ to within three significant figures. Secondly, in Section 14.3, the best algebraic constructions using the direct product of the additive and multiplicative groups of a Galois field and a cyclic group achieve values for the quadratic coefficient that are just below $3/8$. Thirdly, in the same section, an upper bound was established

for Abelian Cayley graphs with this type of construction with a quadratic coefficient of $3/8$.

Similar evidence also exists for the diameter 3 case. Unfortunately, the extremal diameter 3 circulant graphs have only been discovered up to degree 15 (see Appendix D), which is a quite restricted set for determining with confidence whether a cubic polynomial in the degree can provide good estimates for their order and for estimating its leading coefficient. With this proviso, the best fit is obtained with a cubic coefficient of about 0.074, although the residual remains small between 0.070 and 0.080.

The upper bound of $3/8$ for diameter 2 Abelian Cayley graph constructions using direct products of finite fields, that was established by the author [28], was extended for similar constructions to provide an upper bound for any fixed diameter, also by the author. This is included with proof as Observation 3.7 in a jointly authored paper [1]. For fixed diameter k , the upper bound on the leading coefficient in a polynomial of degree k for the order of similarly constructed Abelian Cayley graphs of degree d is given by $(k+1)/(2(k+2)^{k-1})$, so that, for any such graph of order $n(d, k)$ we have

$$n(d, k) \leq \frac{k+1}{2(k+2)^{k-1}} d^k + O(d^{k-1}).$$

For diameter 3, this coefficient evaluates to $4/50$ or 0.080 , which is in good alignment with the fitted cubic above. Thus, for circulant graphs of diameters 2 and 3, we see that the best fitted polynomial of corresponding degree has a leading coefficient lying very close to an upper bound for a category of algebraic constructions giving graphs of largest order. Unfortunately, extremal circulant graphs of diameter 4 and above are known for too few degrees to enable a similar investigation for higher diameters. Nevertheless, extrapolating the diameters 2 and 3 results leads to the following conjecture for any diameter $k \geq 1$.

Conjecture 14.9. *Let $Ext_{circ}(d, k)$ denote the order of an extremal circulant graph of degree d and diameter k . Then for any diameter $k \geq 1$:*

$$\liminf_{d \rightarrow \infty} \frac{Ext_{circ}(d, k)}{d^k} = \limsup_{d \rightarrow \infty} \frac{Ext_{circ}(d, k)}{d^k} = \frac{k+1}{2(k+2)^{k-1}}.$$

The Pott and Zhou constructions [44] are proof that this conjecture is not extendable to Abelian Cayley graphs in general. It might well be that, in time, similar constructions for circulant graphs will also be found that invalidate this conjecture.

DIRECTED AND MIXED CIRCULANT GRAPH FAMILIES OF GIVEN DEGREE

Chapter 15 is the first chapter where the graphs are not undirected. Directed and mixed circulant graphs and graph families are explored for small fixed directed and undirected degree, building on the legacy position described in Section 2.6. As defined in Section 1.2, a directed graph is a graph where all the edges are directed edges, called arcs. A mixed graph is a graph with at least one undirected edge and one arc. Directed and mixed Abelian Cayley graphs have a connection set that is not inverse-closed. In contrast to undirected circulant graphs, it appears that extremal directed and mixed circulant graphs of dimension 3 and above do not belong to graph families with regular diameter classes. An extremal order conjecture is presented for directed and mixed circulant graphs with undirected degree below 4.

As every Abelian Cayley graph is regular, the outdegree of a directed circulant graph is equal to the indegree and is called the *directed degree* of the graph, with symbol z . Then $Ext_{circ}^{dir}(z, k)$ denotes the order of an extremal directed circulant graph of directed degree z and diameter k . Now consider a mixed circulant graph where each vertex has z incident arcs and d incident edges. Then z is the *directed degree* and d the *undirected degree*. The order of an extremal mixed circulant graph of directed degree z , undirected degree d and diameter k is denoted by $Ext_{circ}^{mix}(z, d, k)$. Where the extremal order is unknown, the largest-known order is denoted by $LK_{circ}^{dir}(z, k)$ or $LK_{circ}^{mix}(z, d, k)$ for directed or mixed graphs respectively. As stated in Section 2.5, we define the dimension f of a general circulant graph (one which is undirected, directed or mixed) of directed degree z and undirected degree d to be $f = z + \lfloor d/2 \rfloor$.

In contrast to undirected circulant graphs, any directed or mixed circulant graph on n vertices has rotational symmetry but might not contain any reflexive symmetry. Thus, its automorphism group is either the cyclic group on n elements, C_n , of order n , or contains the cyclic group as a subgroup, and similarly for directed or mixed Abelian Cayley graphs. Therefore, the order of the automorphism group of a directed or mixed Abelian Cayley graph of order n will be a multiple of n . We define this multiple to be the *cyclic index* of the automorphism group, or CI for short.

We begin by establishing a theorem that is useful for the comparison of the orders of extremal and largest-known circulant graph families of the same dimension and directed degree but with even and odd undirected degree. As a consequence, the leading coefficient of the polynomial for a lower bound or an upper bound for odd undirected degree is exactly double the equivalent for even undirected degree of the same dimension and directed degree.

Lemma 15.1. *Let $X(n, z, d, k)$ be a circulant graph of directed degree $z \geq 0$, undirected degree $d \geq 0$, order n and diameter k . Then if the undirected degree d is even, there exists a circulant graph $X'(2n, z, d+1, k+1)$ of directed degree z , undirected degree $d+1$, order $2n$ and diameter $k+1$. Conversely, if the undirected degree d is odd, there exists a circulant graph $X'(n/2, z, d-1, k-1)$ of directed degree z , undirected degree $d-1$, order $n/2$ and diameter $k-1$.*

Proof. The proof of the first part is simply the proof of Theorem 9.3, relaxing the constraint on generators being odd. The second part is similarly proved as for Theorem 9.2. \square

Theorem 15.2. *Let $\mathcal{X}(z, d)$ be a family of circulant graphs of directed degree z and undirected degree d with order $c_f k^f + c_{f-1} k^{f-1} + \dots + c_0$, where $f = z + \lfloor d/2 \rfloor$ is the dimension and k is the diameter. Then if the undirected degree d is even, there exists a circulant graph family $\mathcal{X}'(z, d+1)$ of directed degree z and undirected degree $d+1$ with order $2c_f k^f + O(k^{f-1})$ where k is the diameter. Conversely, if the undirected degree d is odd, there exists a circulant graph family $\mathcal{X}'(z, d-1)$ of directed degree z and undirected degree $d-1$ with order $(c_f/2)k^f + O(k^{f-1})$, where k is the diameter.*

Proof. A direct consequence of Lemma 15.1. \square

15.1 Extremal mixed circulant graphs of dimensions 1 and 2

We now consider mixed circulant graphs, starting with dimension 1: directed degree $z = 1$ and undirected degree $d = 1$. The extremal mixed circulant graph of diameter k is simply the directed cycle graph of order $2k$ with undirected diameters generated by the involution, so that $Ext_{circ}^{mix}(1, 1, k) = 2k$, just 1 short of the upper bound $Upp_{circ}^{mix}(1, 1, k) = 2k + 1$.

For dimension 2 with directed degree $z = 2$, the undirected degree is $d = 1$. Extremal families exist and are defined for three diameter classes, modulo 3. Their order $Ext_{circ}^{mix}(2, 1, k)$, for diameter $k \geq 6$, is given below, along with the upper bound for

comparison:

$$Ext_{circ}^{mix}(2, 1, k) = \begin{cases} (2k^2 + 4k)/3 & \text{for } k \equiv 0 \pmod{3} \\ (2k^2 + 4k)/3 & \text{for } k \equiv 1 \pmod{3} \\ (2k^2 + 4k - 4)/3 & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

$$Upp_{circ}^{mix}(2, 1, k) = k^2 + 2k + 1.$$

There are multiple isomorphism classes for each diameter class. The existence and extremality of graph families with these orders have been proved by Dalfó, Fiol and López [8]. One family from each diameter class is shown in Table 15.1 as an example, along with a generating set. Properties of all isomorphism classes of graphs up to diameter 16 are given in Appendix G.

Table 15.1: Generating sets for extremal mixed circulant graph families of directed degree 2 and undirected degree 1 for diameter $k \geq 6$

Family	Diameter class	Generating set*		Odd girth	Girth	Maximal levels	Aut group CI [†]
M2-1:0A	0 (mod 3)	1	$k + 1$	$(2k + 3)/3$	$(2k + 3)/3$	$(2k - 3)/3$	1
M2-1:1A	1 (mod 6)	1	$k + 2$	bipartite	$(2k + 4)/3$	$(2k - 2)/3$	1
	4 (mod 6)	1	$k + 2$	$k + 1$	$(2k + 4)/3$	$(2k - 2)/3$	1
M2-1:2A	2 (mod 6)	1	k	$(5k - 1)/3$	$(2k + 2)/3$	$(2k - 4)/3$	1
	5 (mod 6)	1	k	bipartite	$(2k + 2)/3$	$(2k - 4)/3$	1

* plus the involution

[†] cyclic index of the automorphism group

For dimension 2 with directed degree $z = 1$, the undirected degree may be 2 or 3. In both cases largest-known families exist and are defined for three diameter classes, modulo 3. For undirected degree $d = 2$, their order $LK_{circ}^{mix}(1, 2, k)$, for diameter $k \geq 3$ is given below, together with the upper bound:

$$LK_{circ}^{mix}(1, 2, k) = \begin{cases} (2k^2 + 6k + 3)/3 & \text{for } k \equiv 0 \pmod{3} \\ (2k^2 + 6k + 4)/3 & \text{for } k \equiv 1 \pmod{3} \\ (2k^2 + 6k + 4)/3 & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

$$Upp_{circ}^{mix}(1, 2, k) = k^2 + 2k + 1.$$

For diameter class $k \equiv 0 \pmod{3}$, there is one isomorphism class for each diameter, forming a single graph family M1-2:0. For the other two diameter classes, there are a varying number of isomorphism classes depending on the diameter, forming a range of families with different diameter period. The examples described, M1-2:1A and

M1-2:2A, exist for all diameters in their class. Generating sets are shown in Table 15.2.

Table 15.2: Generating sets for largest-known mixed circulant graph families of directed degree 1 and undirected degree 2 for diameter $k \geq 3$

Family	Diameter class	Generating set		Odd girth	Maximal levels	Aut group CI [†]
		Directed	Undirected			
M1-2:0	0 (mod 3)	1	$(2k+3)/3$	$(4k+3)/3$	$2k/3$	1
M1-2:1A	1 (mod 3)	$k+1$	$(k+2)/3$	$(4k+5)/3$	$(2k+1)/3$	2
M1-2:2A	2 (mod 3)	$k+2$	$(k+1)/3$	$(4k+7)/3$	$(2k-1)/3$	2

[†] cyclic index of the automorphism group

For undirected degree $d = 3$, the order of the largest-known families, $LK_{circ}^{mix}(1, 3, k)$, for diameter $k \geq 3$ is given below, with the upper bound:

$$LK_{circ}^{mix}(1, 3, k) = \begin{cases} (4k^2 + 8k)/3 & \text{for } k \equiv 0 \pmod{3} \\ (4k^2 + 8k)/3 & \text{for } k \equiv 1 \pmod{3} \\ (4k^2 + 8k - 8)/3 & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

$$Upp_{circ}^{mix}(1, 3, k) = 2k^2 + 2k + 1.$$

For diameter classes $k \equiv 0$ and 1, there is one isomorphism class for each diameter, forming single families M1-3:0 and M1-3:1. For diameter class 2, there are two isomorphism classes for each diameter $k \geq 5$, forming families M1-3:2A and M1-3:2B, as well as two additional classes for diameter 5. The existence and extremality of graph families with these orders was demonstrated using tiling constructions for $k \equiv 0$ and 1 (mod 3) by Dalfó, Fiol, López and Ryan [9]. Generating sets are shown in Table 15.3.

Table 15.3: Generating sets for largest-known mixed circulant graph families of directed degree 1 and undirected degree 3 for diameter $k \geq 3$

Family	Diameter class	Generating set		Odd girth	Maximal levels	Aut group CI [†]
		Directed	Undirected			
M1-3:0	0 (mod 3)	1	$(2k+3)/3$	$(4k+3)/3$	$2k/3$	1
M1-3:1	1 (mod 3)	1	$(2k+1)/3$	$(4k+5)/3$	$(2k-2)/3$	1
M1-3:2A	2 (mod 3)	1	$(2k-1)/3$	$(4k+7)/3$	$(2k-4)/3$	1
M1-3:2B	2 (mod 3)	1	$(2k+5)/3$	$(4k+1)/3$	$(2k-1)/3$	1

[†] cyclic index of the automorphism group

15.2 Extremal directed circulant graphs of dimension 3

For dimension 3, Wong and Copperfield's upper bound for directed circulant graphs gives $Upp_{circ}^{mix}(3, 0, k) = (k^3 + 6k^2 + 11k + 6)/6$ for any diameter k . Sharper lower and upper bounds for the cubic coefficient are $1/16$ and $3/25$ (see Section 2.5).

Largest-known directed circulant graphs of directed degree 3 have been discovered up to diameter 48: as far as diameter 17 by Fiduccia, Forcada and Zito [14], up to 43 by Dougherty and Faber [10] and the final five by the author (see Appendix G). They have been verified to be extremal up to diameter 39 and are conjectured extremal to 48. They do not have order determined by polynomials defined for a set of diameter classes and so do not belong to graph families. Instead, the sequence of extremal orders appears chaotic or random. If there is indeed any underlying pattern or structure to this sequence, then it emerges only at a higher diameter than 48.

However, it is conjectured that there is no such structure and that the sequence, for increasing diameter, of orders of extremal directed circulant graphs of directed degree 3 is chaotic throughout.

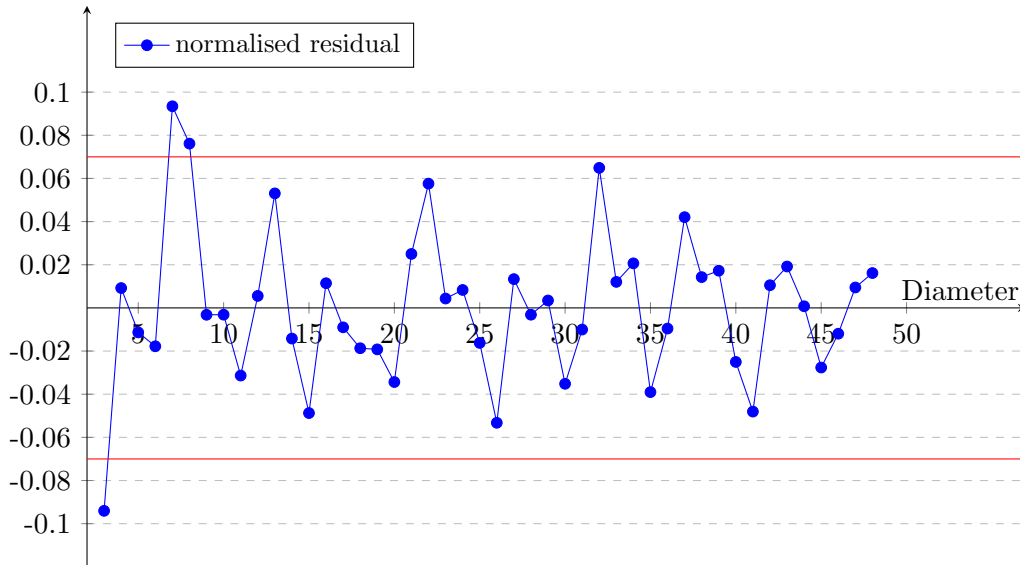
Conjecture 15.3. *There is no finite set of diameter classes $\{K_1, \dots, K_g\}$ where for any diameter k , $k \in K_i$ for some i , and no set of directed circulant graph families $\{F_1, \dots, F_g\}$ defined for each diameter class and with order $n_1(k), \dots, n_g(k)$ respectively, where $n_i(k)$ is a cubic polynomial, such that for any i and any $k \in K_i$, $Ext_{circ}^{dir}(3, k) = n_i(k)$.*

Notwithstanding this chaotic behaviour, it is possible to define a cubic polynomial $\hat{n}(k)$ in the diameter k that achieves a least-squares fit to the extremal orders up to diameter 48. This cubic, to four significant figures, is

$\hat{n}(k) = 0.08341k^3 + 0.7065k^2 + 2.058 + 1.796$. The question arises whether the leading coefficient, 0.08341, is a statistical approximation to an underlying actual coefficient that can be expressed as a simple fraction, similar to the leading coefficients 1 and $1/3$ for directed degrees 1 and 2. The value 0.08341 lies relatively close to the unit fraction $1/12$ (0.08333 to five decimals), and no other simple fraction with denominator below 539 lies closer ($45/539$ is closer). This justifies exploring $1/12$ as the leading coefficient of the underlying cubic best fit for all diameters. Accepting an additional total least-squares residual of less than 0.01%, we may instead fit a cubic with leading coefficient of $1/12$: $\hat{n}(k) = (1/12)k^3 + 0.7125k^2 + 1.936k + 2.377$.

Normalising the residual term for each diameter by dividing it by k^2 enables the residual to be evaluated as a component of the quadratic coefficient. The normalised residuals are shown in Figure 15.1.

Figure 15.1: Order of extremal directed circulant graphs of directed degree 3, up to diameter 48: least-squares residual divided by the diameter squared



Above a threshold diameter of 8, the absolute value of the normalised residual term lies below 0.07 for all diameters up to 48. Moreover, it appears that the range is not increasing with diameter but remains relatively constant, as indicated by the parallel red lines. If this remains true for all higher diameters, then we have the following conjecture.

Conjecture 15.4. *The extremal order of a directed circulant graph of directed degree 3 and diameter k , $Ext_{circ}^{dir}(3, k)$ is bounded below and above:*

$$(1/12)k^3 + 0.6k^2 + O(k) \leq Ext_{circ}^{dir}(3, k) \leq (1/12)k^3 + 0.8k^2 + O(k).$$

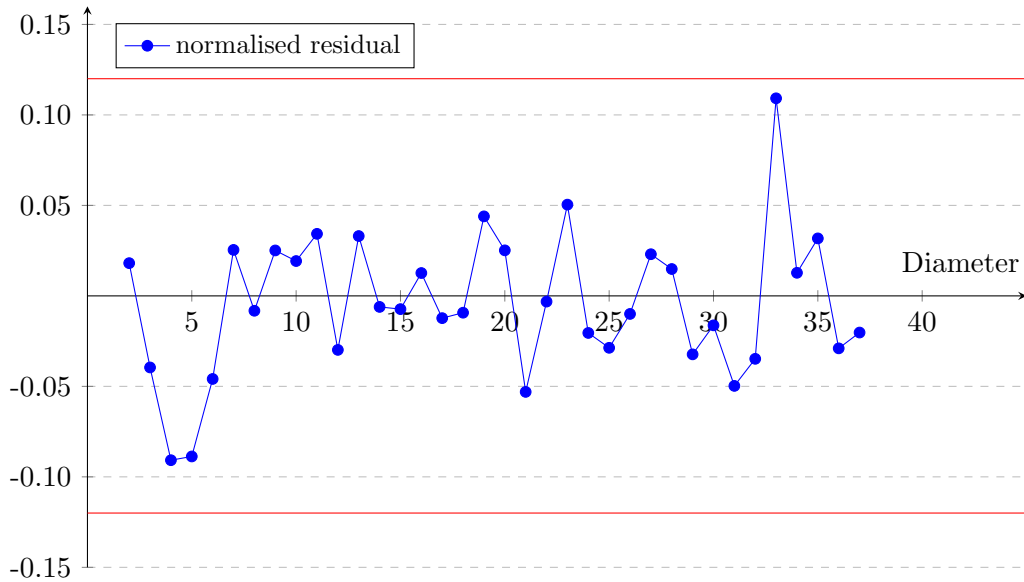
In particular, this implies that $Ext_{circ}^{dir}(3, k) = \frac{1}{12}k^3 + O(k^2)$.

15.3 Extremal mixed circulant graphs of dimension 3

Mixed circulant graphs of dimension 3 may have directed degree of 1, 2 or 3. It emerges that the same chaotic behaviour displayed by the order of directed circulant graphs of dimension 3 is apparent for mixed circulant graphs of all degree parameter sets (z, d) with dimension $f = 3$. In each case, a polynomial in the diameter has been fitted to the graph orders. The leading coefficient has been adjusted to the nearest simple fraction with low denominator. In each case, the magnitude of the normalised residuals appears to vary within a range that remains broadly constant with increasing diameter, above a small diameter threshold. This mirrors what is observed for directed graphs of dimension 3 above.

We first consider graphs of directed degree $z = 3$ along with the involution ($d = 1$). Largest-known circulant graphs that are conjectured to be extremal have been discovered up to diameter 37 (see Appendix G). A least squares fit by a cubic in the diameter k yields the following: $\hat{n} = 0.16673k^3 + 0.8308k^2 + 5.147k - 3.139$. The unit fraction $1/6$ is close to 0.16673 and closer than any other simple fraction with denominator below 1409 (235/1409 is closer). With an additional total least-squares residual of 0.01%, we may instead fit a cubic with leading coefficient of $1/6$: $\hat{n} = (1/6)k^3 + 0.8343k^2 + 5.090k - 2.923$. As before, above a threshold diameter of 5, the normalised residuals appear to vary within a range that remains constant with increasing diameter, bounded by ± 0.06 with the exception of an outlier at diameter 33 that stretches the bound to ± 0.12 , see Figure 15.2.

Figure 15.2: Order of extremal mixed circulant graphs of directed degree 3 and undirected degree 1, up to diameter 37: least-squares residual divided by the diameter squared



This supports the following conjecture.

Conjecture 15.5. *The extremal order of a mixed circulant graph of directed degree 3 and undirected degree 1, for diameter $k \geq 2$, $Ext_{circ}^{mix}(3, 1, k)$ is bounded below and above:*

$$(1/6)k^3 + 0.7k^2 + O(k) \leq Ext_{circ}^{mix}(3, 1, k) \leq (1/6)k^3 + 1.0k^2 + O(k)$$

implying that $Ext_{circ}^{mix}(3, 1, k) = \frac{1}{6}k^3 + O(k^2)$.

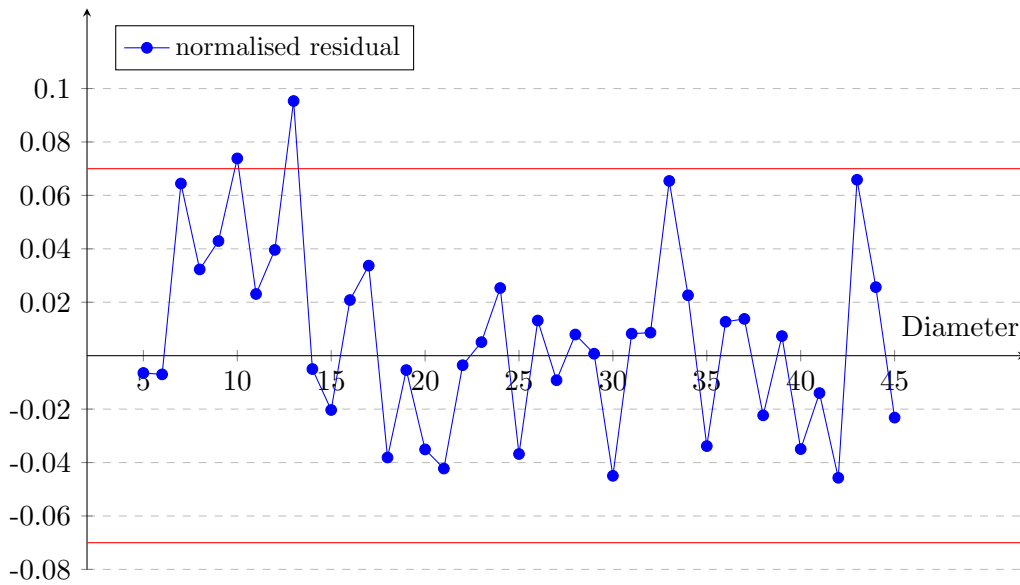
For dimension 3 and directed degree $z = 2$, mixed circulant graphs have undirected degree of either 2 or 3. For undirected degree $d = 2$, extremal and largest-known graphs have been discovered up to diameter 45 (see Appendix G). A least-squares fit

by a cubic in the diameter k yields the following:

$\hat{n}(k) = 0.1674k^3 + 1.201k^2 + 1.115k + 8.965$. In this case, the unit fraction $1/6$ is close to 0.1674, and closer than any other simple fraction with denominator below 125.

With an additional total least-squares residual of 0.2%, we may instead fit a cubic with leading coefficient of $1/6$: $\hat{n}(k) = (1/6)k^3 + 1.250k^2 + 0.1724k + 13.21$. Above a threshold diameter of 13, the normalised residuals appear to vary within a range that remains constant with increasing diameter, bounded by ± 0.07 , see Figure 15.3.

Figure 15.3: Order of extremal mixed circulant graphs of directed degree 2 and undirected degree 2, up to diameter 45: least-squares residual divided by the diameter squared



This supports the following conjecture.

Conjecture 15.6. *The extremal order of a mixed circulant graph of directed degree 2, undirected degree 2, and diameter $k \geq 14$, $Ext_{circ}^{mix}(2, 2, k)$ is bounded below and above:*

$$(1/6)k^3 + 1.1k^2 + O(k) \leq Ext_{circ}^{mix}(2, 2, k) \leq (1/6)k^3 + 1.4k^2 + O(k)$$

implying that $Ext_{circ}^{mix}(2, 2, k) = \frac{1}{6}k^3 + O(k^2)$.

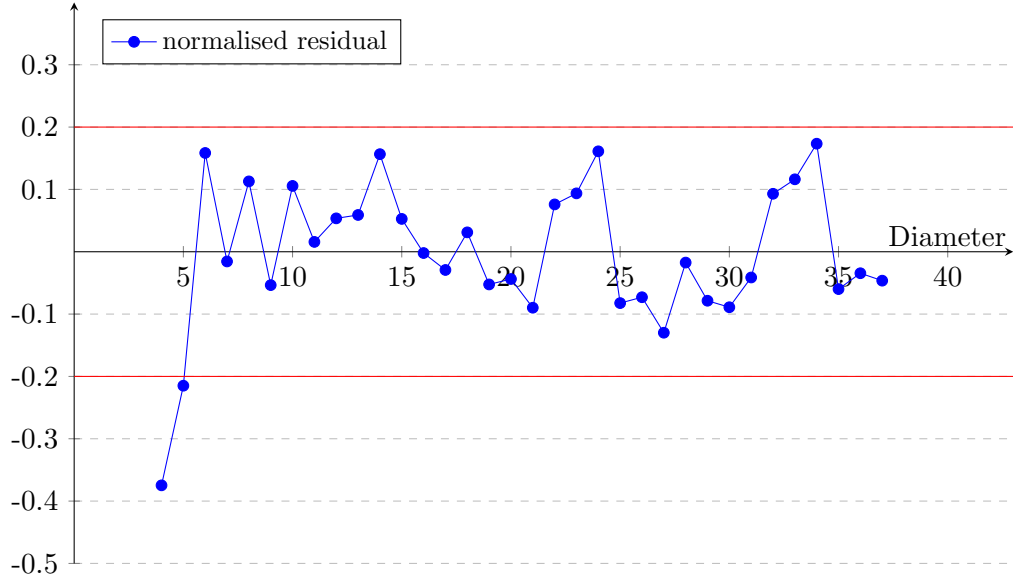
For directed degree 2 and undirected degree 3, extremal and largest-known graphs have been discovered up to diameter 37 (see Appendix G). A least-squares fit by a cubic in the diameter k gives the following:

$\hat{n}(k) = 0.3373k^3 + 1.539k^2 + 4.880k - 2.596$. The unit fraction $1/3$ is close to 0.3373, and closer than any other simple fraction with denominator below 44. With an extra total residual of 0.4%, we may instead fit a cubic with leading coefficient $1/3$:

$\hat{n}(k) = (1/3)k^3 + 1.773k^2 + 1.088k + 11.95$. Above a threshold diameter of 5, the

normalised residuals vary within a range that remains constant with increasing diameter, bounded by ± 0.20 , see Figure 15.4.

Figure 15.4: Order of extremal mixed circulant graphs of directed degree 2 and undirected degree 3, up to diameter 37: least-squares residual divided by the diameter squared



This supports the following conjecture.

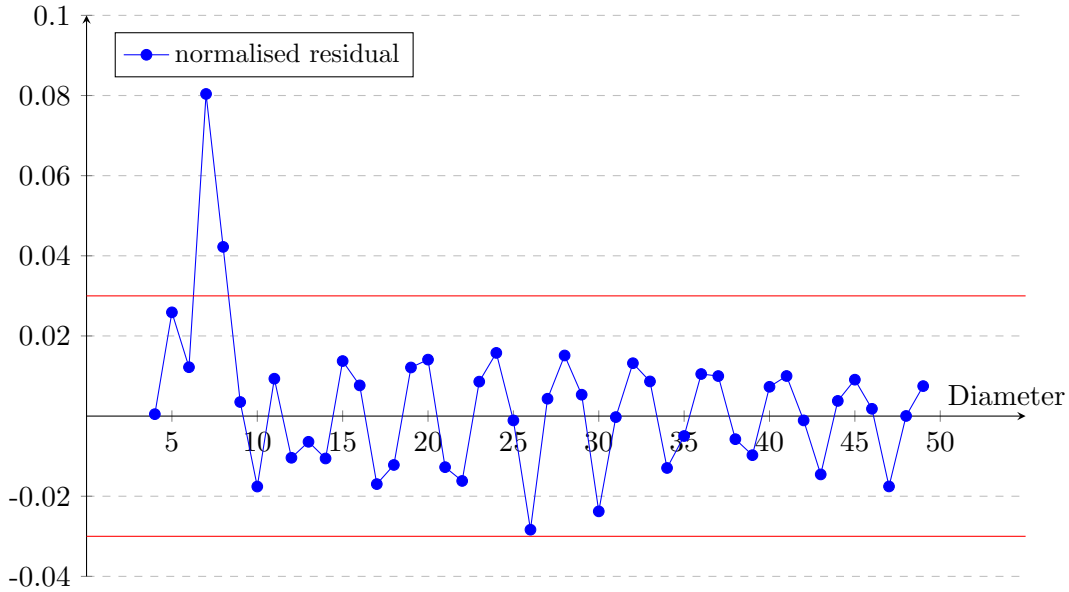
Conjecture 15.7. *The extremal order of mixed circulant graphs of directed degree 2, undirected degree 3, and diameter $k \geq 6$, $Ext_{circ}^{mix}(2, 3, k)$ is bounded below and above:*

$$(1/3)k^3 + 1.5k^2 + O(k) \leq Ext_{circ}^{mix}(2, 3, k) \leq (1/3)k^3 + 2.0k^2 + O(k)$$

implying that $Ext_{circ}^{mix}(2, 3, k) = \frac{1}{3}k^3 + O(k^2)$.

For dimension 3 and directed degree $z = 1$, mixed circulant graphs have undirected degree of either 4 or 5. For undirected degree $d = 4$, extremal and largest-known graphs have been discovered up to 49 (see Appendix G). A least squares fit by a cubic in the diameter k yields the following: $\hat{n}(k) = 0.40735k^3 + 1.817k^2 + 1.103k + 3.579$. The simple fraction $11/27$ is close to 0.40735 and closer than any other simple fraction with denominator below 328. With an additional residual of less than 0.02%, a cubic with leading coefficient $11/27$ may be substituted: $\hat{n}(k) = (11/27)k^3 + 1.813k^2 + 1.194k + 3.138$. Dividing the residual term for each diameter by k^2 normalises the residual for comparison with the quadratic coefficient. This is shown in Figure 15.5.

Figure 15.5: Order of largest-known mixed circulant graphs of directed degree 1 and undirected degree 4, up to diameter 49: least-squares residual divided by the diameter squared



Above a threshold diameter of 8, the absolute value of the normalised residual term remains below 0.03. If the residual remains within this bound for the extremal order for all higher diameters, then we have the following conjecture.

Conjecture 15.8. *The extremal order of a mixed circulant graph of directed degree 1, undirected degree 4 and arbitrary diameter $k \geq 9$, $Ext_{circ}^{mix}(1, 4, k)$ is bounded below and above:*

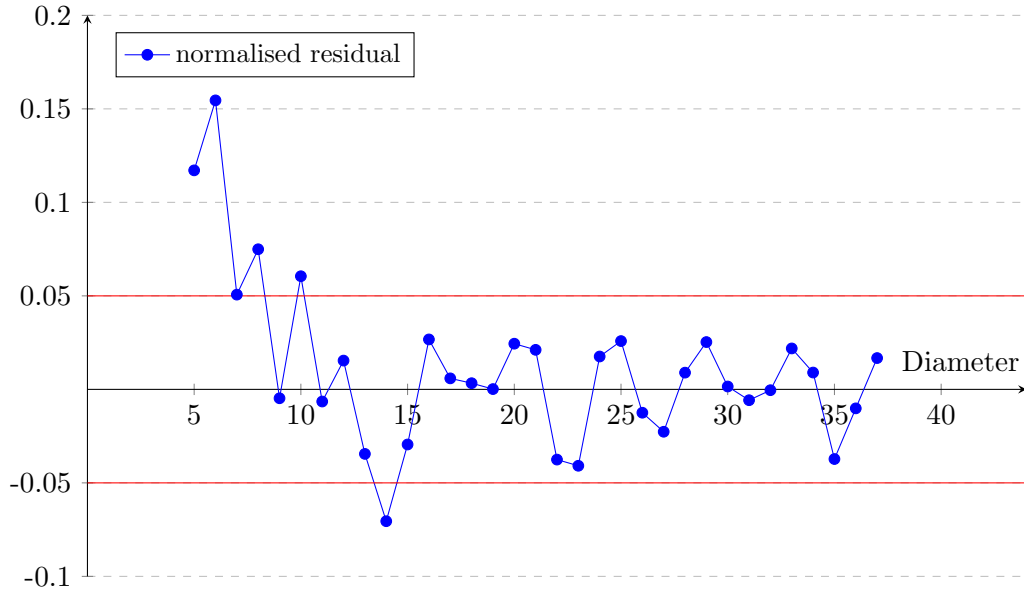
$$(11/27)k^3 + 1.7k^2 + O(k) \leq Ext_{circ}^{mix}(1, 4, k) \leq (11/27)k^3 + 1.9k^2 + O(k).$$

In particular, this implies that $Ext_{circ}^{mix}(1, 4, k) = \frac{11}{27}k^3 + O(k^2)$.

For directed degree 1 and undirected degree 5, extremal and largest-known graphs have been discovered up to diameter 37 (see Appendix G). A least-squares fit by a cubic in the diameter k gives the following:

$\hat{n}(k) = 0.8146k^3 + 2.203k^2 - 1.073k + 9.582$. The simple fraction $22/27$ is very close to 0.8146 and closer than any other simple fraction with denominator below 96. With an extra total residual of 0.02%, we may instead fit a cubic with leading coefficient $22/27$: $\hat{n}(k) = (22/27)k^3 + 2.192k^2 - 0.8919k + 8.888$. Above a threshold diameter of 14, the normalised residuals vary within a range of ± 0.05 , see Figure 15.6.

Figure 15.6: Order of largest-known mixed circulant graphs of directed degree 1 and undirected degree 5, up to diameter 37: least-squares residual divided by the diameter squared



This supports the following conjecture:

Conjecture 15.9. *The extremal order of mixed circulant graphs of directed degree 1, undirected degree 5, and diameter $k \geq 15$, $Ext_{circ}^{mix}(1, 5, k)$ is bounded below and above:*

$$(22/27)k^3 + 2.1k^2 + O(k) \leq Ext_{circ}^{mix}(1, 5, k) \leq (22/27)k^3 + 2.3k^2 + O(k)$$

implying that $Ext_{circ}^{mix}(1, 5, k) = \frac{22}{27}k^3 + O(k^2)$.

15.4 Conjectured order of extremal directed & mixed circulant graphs

In previous chapters, we have seen that extremal and largest-known undirected circulant graphs of degree up to 20 and beyond, above low diameter thresholds, all belong to families defined for a set of diameter classes. Also, as we have just seen, this is true for the extremal directed and mixed circulant graphs of dimensions 1 and 2. However, this fails for dimension 3, as far as checked.

For each of the various combinations of directed and undirected degrees discussed so far in this chapter, the order of verified and conjectured extremal circulant graphs increases with diameter according to a fitted polynomial in the diameter k of degree equal to the dimension f with a residual of $O(k^{f-1})$. Where the directed degree is 0 or the dimension is below 3, the residual displays a regularity with diameter periodicity equal to the dimension. For these degree combinations, families of graphs

can be defined for all diameter classes, with order and generating sets defined precisely by polynomials in the diameter. On the other hand, when the directed degree is at least 1 and the dimension is 3, the residual has no such periodic regularity, but instead displays apparent chaotic behaviour, so that extremal graph families cannot be defined. This is covered later by Conjecture 15.11.

Despite the distinction between these two categories of circulant graphs, regular and chaotic, determined by the combination of directed and undirected degree, they share the existence of a polynomial in the diameter of degree equal to the dimension as a least-squares fit to the order of the extremal graphs for each combination. For directed and mixed circulant graphs of dimension 3, the fitted cubics for each degree combination are summarised in Table 15.4.

Table 15.4: Largest-known directed and mixed circulant graphs of dimension 3

Directed/ undirected degree		Largest diameter included	Coefficients of least-squares fit (1)				Inferred leading coefficient		Extra residual
z	d		k^3	k^2	k	Constant	Maximum residual (2)	Coeff	
3	0	48	0.08341	0.7065	2.058	1.796	0.07	1/12	0.01%
3	1	37	0.16673	0.8308	5.147	-3.139	0.12	1/6	0.01%
2	2	45	0.1674	1.201	1.115	8.965	0.07	1/6	0.2%
2	3	37	0.3373	1.539	4.880	-2.596	0.20	1/3	0.4%
1	4	49	0.40735	1.817	1.103	3.579	0.03	11/27	0.01%
1	5	37	0.8146	2.203	-1.073	9.583	0.05	22/27	0.02%

(1) to four significant figures

(2) normalised by dividing by the diameter squared,
maximum for diameters above some low threshold

In each case, the simple fraction inferred as the leading coefficient is very close to the fitted value, and the extra total residual from substituting the inferred coefficients is very low. This provides some confidence that these values are correct.

The leading coefficients of the general circulant upper bound for selected directed and undirected degree combinations are presented in Table 15.5.

Table 15.5: General circulant graph upper bound $Upp_{circ}^{mix}(z, d, k)$:
leading coefficient, $2^{\lfloor (d+1)/2 \rfloor} / f!$, where dimension $f = z + \lfloor d/2 \rfloor$

Directed degree z	Undirected degree d									
	0	1	2	3	4	5	6	7	8	9
0	-	2	2	4	2	4	4/3	8/3	2/3	4/3
1	1	2	1	2	2/3	4/3	1/3	2/3	2/15	4/15
2	1/2	1	1/3	2/3	1/6	1/3	1/15	2/15	1/45	2/45
3	1/6	1/3	1/12	1/6	1/30	1/15	1/90	1/45	1/315	2/315

The inferred leading coefficients of least-squares fitted polynomials for the order of extremal and largest-known circulant graphs are presented in Table 15.6 for the cases considered above. The regular cases, where the graphs belong to families, are shown in bold text. For the others, the residual is chaotic and so the coefficients have a range of uncertainty.

Table 15.6: Extremal general circulant graphs, least-squares fitted polynomial: leading coefficient

Directed degree z	Undirected degree d									
	0	1	2	3	4	5	6	7	8	9
0	-	2	2	4	2	4	32/27	64/27	1/2	1
1	1	2	2/3	4/3	11/27	22/27				
2	1/3	2/3	1/6	1/3						
3	1/12	1/6								

Bold indicates exact fit for largest-known families in each diameter class

The second column in Table 15.5, $d = 0$, is for the directed circulant graphs. They are defined by the formula $1/z!$ as established by Wong and Copperfield (see Section 2.5), which is equal to $1/f!$ in this case. By inspection of the first column in Table 15.6, it emerges that all values may be described by a similar formula: $2/(f + 1)!$. This formula applies equally to the first two values, that describe the order of graph families, and to the third of these values, that relates to a chaotic sequence of orders of graphs that do not belong to families. Consistent with Theorem 15.2, the values in both third columns, $d = 1$, are equal to twice the second.

The fourth columns are for degree combinations with undirected degree $d = 2$. For the upper bound, their formula is $2/f!$. For the extremal mixed circulant graphs with undirected degree $d = 2$, we only have three data points, but they indicate a formula of $4/(f + 1)!$, again valid for graph families and for chaotic sequences. As expected, the values in both fifth columns are double the fourth.

Considering the leading coefficients for all degree combinations with undirected degree $d \leq 3$, in all 13 cases they can be described by a single expression, $2^{\lfloor (d+3)/2 \rfloor} / (f + 1)!$. Moreover, this is a simple multiple, $2/(f + 1)$, of the upper bound leading coefficient. The fact that this expression satisfies all 13 cases without exception lends weight to the validity of the inferred leading coefficient for each individual chaotic case. This also provides some confidence in extrapolating its applicability to all higher directed degrees, for undirected degree $d \leq 3$. This leads to the following conjecture.

Conjecture 15.10. *Extremal Order Conjecture for directed circulant graphs and some mixed graphs. For any dimension f and directed degree z with undirected degree $d \leq 3$, there exists a polynomial $\hat{n}(z, d, k)$ of degree f in the diameter k that is a*

least-squares fit to the order $Ext_{circ}^{mix}(z, d, k)$ of the extremal circulant graphs of every diameter with $\hat{n}(z, d, k) = \frac{2^{\lfloor (d+3)/2 \rfloor}}{(f+1)!} k^f + O(k^{f-1})$, such that $Ext_{circ}^{mix}(z, d, k) - \hat{n}(z, d, k) = O(k^{f-1})$.

In Section 2.5, an upper bound was defined for all degree combinations that is also a polynomial in the diameter k of degree equal to the dimension f and with leading term $(2^{\lfloor (d+1)/2 \rfloor} / f!) k^f$. The question is whether there exists a corresponding leading term that is common to all least-squares fitted polynomials for extremal orders. From the Extremal Order Conjecture 3.1, we already have a term for undirected graphs, $(2^{d-1} / f^f) k^f$. And from the Extremal Order Conjecture for directed and mixed circulant graphs up to undirected degree 3, Conjecture 15.10, we have leading term $(2^{\lfloor (d+3)/2 \rfloor} / (f+1)!)$. What remains to be addressed are extremal mixed circulant graphs of undirected degree $d \geq 4$.

In the fifth column of Table 15.6, $d = 4$, the value of $11/27$ for directed degree $z = 1$, undirected degree $d = 4$ (dimension $f = 3$) appears at first sight to be anomalous. In particular, it is not consistent with Conjecture 15.10. However, $11/27$ shares denominator with the coefficient, $32/27$, for the undirected case $z = 0$, $d = 6$, which has the same dimension $f = 3$. The question therefore arises whether there exists a relation between the leading coefficients of the polynomials fitted to extremal circulant graphs of the same dimension f , valid equally for undirected, directed and mixed graphs. We will consider only graphs with even undirected degree. Any result can be directly translated to odd undirected degrees by invoking Theorem 15.2. For undirected even-degree graphs, the leading coefficient simplifies to $2^{d/2+1} / f^f$. For directed and mixed even-degree graphs up to undirected degree 2 (so just $d = 0$ and $d = 2$), the leading coefficient is $2^{d/2+1} / (f+1)!$. Thus, a reasonable initial conjecture would be that for circulant graphs of any degree combination (undirected, directed or mixed) the leading coefficient of the fitted order polynomial in the diameter may be expressed as an integer multiple of $2^{d/2+1} / ((f+1)! f^{f-1})$. So for directed graphs, this multiple is f^{f-1} , and for undirected graphs, $2^{f-2} (f+1)(f-1)!$. The conjectured multiples are shown in Table 15.7.

Reading up the skew diagonals of Table 15.7 for each dimension f gives the following sequences: $f = 1$: 1 1; $f = 2$: 2 2 3; $f = 3$: 9 9 11 16; $f = 4$: 64 64 x y 120, for unknown x , y . It is tempting to assume that, for higher dimensions, the multiple will step monotonically from the directed degree value to the undirected degree. These steps would be positive increments up to dimension 10. However, above dimension 10, the directed degree multiple exceeds the undirected degree value, and so these steps would be negative.

Table 15.7: Conjectured leading coefficient of the fitted order polynomial as a multiple of $2^{d/2+1}/((f+1)!f^{f-1})$ for dimension f and undirected degree d

Directed degree z	Undirected degree d					
	0	2	4	6	8	10
0		1	3	16	120	1152
1	1	2	11			
2	2	9				
3	9	64				
4	64	625				
5	625	7776				

Now we consider the sequence for dimension 4. The two steps from 9 to 16 for dimension 3 are 2 and 5, the second being just over a doubling of the first. For dimension 4, the sequence increases from 64 to 120, a total of 56 in three steps. In this case, a doubling of each step fits nicely, as 56 is divisible by 7. Increases of 8, 16 and 32 would give a sequence of (64, 64, 72, 88, 120). There is no reason to believe the values 72 and 88 are correct; they are just an initial hypothesis. This tentative sequence corresponds to the leading coefficients shown in Table 15.8.

Table 15.8: Conjectured leading coefficients of extremal dimension 4 graph families given by tentative multiples sequence (64, 64, 72, 88, 120)

Graph type	Directed degree z	Undirected degree d	Multiple in sequence	Divisor	Leading coefficient	Corresponding conjecture
Directed	4	0	64	3840	1/60	15.10
Mixed	3	2	64	1920	1/30	15.10
Mixed	2	4	72	960	3/40	-
Mixed	1	6	88	480	11/60	-
Undirected	0	8	120	240	1/2	3.1

15.5 Extremal directed & mixed circulant graphs above dimension 3

For dimensions 4 and 5, because of the greater number of generators and the larger order of the graphs, it has only been possible to run computer searches for extremal graphs up to much lower maximum diameters. Therefore, the fitted polynomials have a much higher level of uncertainty, including reduced confidence in identifying the most appropriate simple fraction for the leading coefficient. There are risks associated with including these results in the data set used to try to establish a common expression for the leading coefficient of polynomials for the order of extremal general circulant graphs of any given directed and undirected degree. Instead, these results

are used only to test the validity of Conjecture 15.10 and the tentative dimension 4 sequence presented in Table 15.8.

For directed circulant graphs of directed degree 4, the quartic coefficients of the lower and upper bounds for the extremal order are $1/256$ and $1/24$. Extremal and largest-known directed-degree 4 circulant graphs have been discovered by the author up to diameter 22 (see Appendix G). All are conjectured to be extremal. As with the dimension 3 case, they do not have order determined by polynomials defined for a set of diameter classes, but instead display chaotic behaviour.

Conjecture 15.11. *For dimension $f \geq 3$, directed degree $z \geq 1$, and undirected degree d , there is no finite set of diameter classes $\{K_1, \dots, K_g\}$ where for any diameter k , $k \in K_i$ for some i , and no set of directed or mixed circulant graph families $\{F_1, \dots, F_g\}$ defined for each diameter class and with order $n_1(k), \dots, n_g(k)$ respectively, where $n_i(k)$ is a polynomial of degree f , such that for any i and any $k \in K_i$, $\text{Ext}_{\text{circ}}^{\text{mix}}(z, d, k) = n_i(k)$ (or $\text{Ext}_{\text{circ}}^{\text{dir}}(z, k)$, in case $d = 0$).*

Unfortunately, a dataset up to only diameter 22 is insufficient for the fitted order polynomial to have stabilised. Every additional result causes the estimated quartic coefficient to jump significantly, typically oscillating around the likely true value. So, fitting a quartic to the full set of results up to diameter 22 gives a quartic coefficient of 0.01865, about $1/54$, whereas up to diameter 21 gives 0.00859, about $1/116$. These values span the value of $1/60$ from Conjecture 15.10, providing some support for the conjecture. The least-squares fit up to diameter 22 by a quartic with leading coefficient $1/60$ gives order polynomial:

$\hat{n}(k) = (1/60)k^4 + 0.0941k^3 + 3.390k^2 - 8.147k + 15.07$. Dividing the residual for each diameter by k^3 normalises the residual as an addition to the cubic coefficient. This is shown in Figure 15.7.

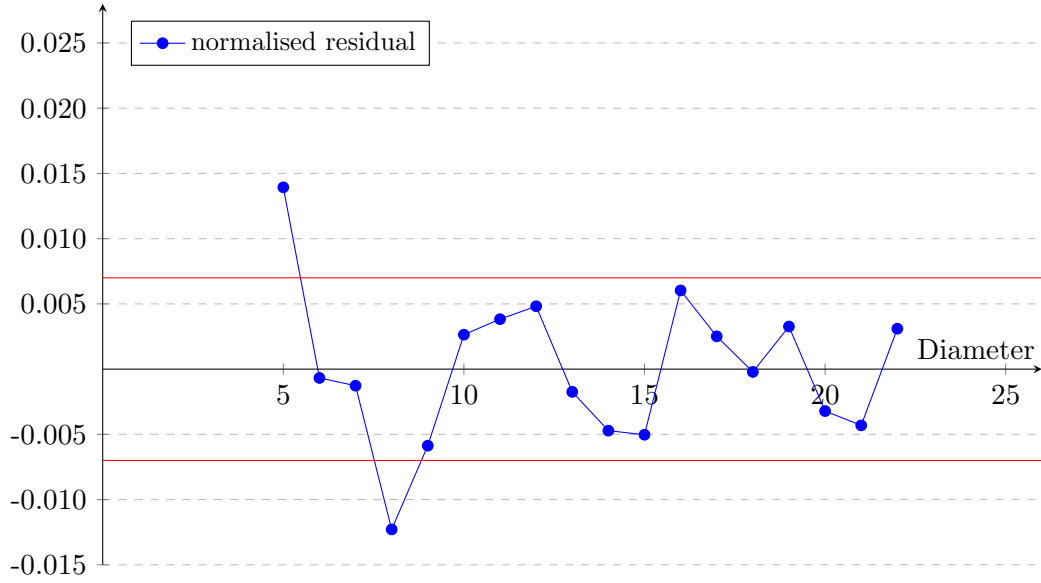
Above diameter 8, the absolute value of the normalised residual lies below 0.007. On the assumption this remains true for all higher diameters, then we have the following conjecture.

Conjecture 15.12. *The extremal order of a directed-degree 4 circulant graph of diameter k , $\text{Ext}_{\text{circ}}^{\text{dir}}(4, k)$ is bounded below and above:*

$$(1/60)k^4 + 0.08k^3 + O(k^2) \leq \text{Ext}_{\text{circ}}^{\text{dir}}(4, k) \leq (1/60)k^4 + 0.11k^3 + O(k^2)$$

implying that $\text{Ext}_{\text{circ}}^{\text{dir}}(4, k) = \frac{1}{60}k^4 + O(k^3)$.

Figure 15.7: Order of extremal and largest-known directed circulant graphs of directed degree 4, up to diameter 22: least-squares residual divided by the diameter cubed



For dimension 5, the quintic coefficients of the lower and upper bounds for the extremal order are $1/3125$ and $1/120$, a very wide range. If Conjecture 15.10 is correct, then the best-fit quintic polynomial for the extremal order would have leading coefficient $1/360$. Extremal and largest-known circulant graphs have been discovered only up to diameter 9 (see Appendix G). This is insufficient to confirm $1/360$ with any confidence, especially as the small magnitude of this coefficient limits the contribution of the quintic term at low diameters. For the full set of results up to diameter 9, the fitted polynomial has a quintic coefficient of 0.003205, about $1/312$. Whereas this lies relatively close to $1/360$, this apparent accuracy may be spurious as the polynomial fit up to diameter 8 has a negative quintic coefficient. The best that can be inferred is that the available data does not contradict an assumption of $1/360$.

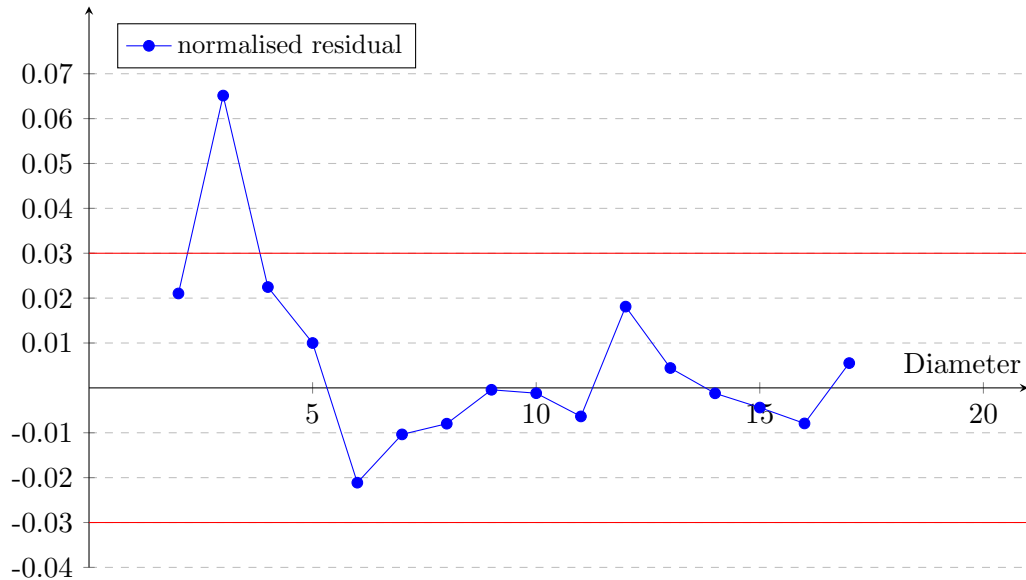
Mixed circulant graphs of dimension 4 have only been investigated for three degree combinations of directed degree z and undirected degree d : $(z, d) = (3, 2), (2, 4), (1, 6)$.

For directed degree 3 and undirected degree 2, extremal and largest-known graphs have been discovered up to diameter 17 (see Appendix G). According to Conjecture 15.10, the quartic coefficient of the fitted order polynomial is $1/30$. The values obtained by best fits up to diameters 16 and 17 are 0.01258 and 0.04801 respectively, approximated by the unit fractions $1/79$ and $1/21$, which is consistent with the conjecture. The least-squares fit up to diameter 17 by a quartic with leading coefficient $1/30$ gives order polynomial:

$$\hat{n}(k) = (1/30)k^4 + 0.1928k^3 + 4.387k^2 - 7.356k + 11.92. \text{ For dimension 4, the residual}$$

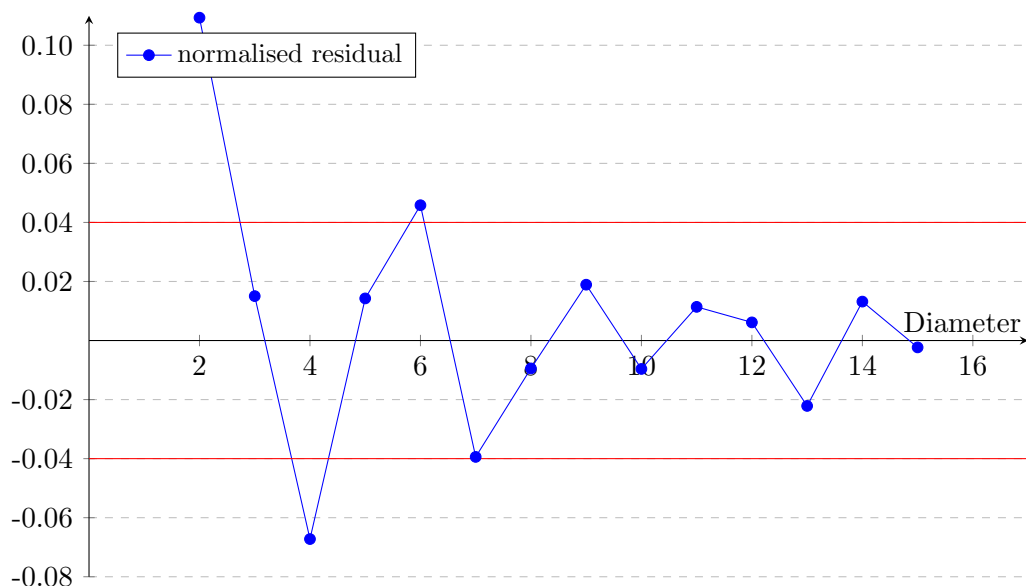
terms are normalised by dividing them by k^3 . Above a threshold diameter of 3, the normalised residuals vary within a range of ± 0.03 , see Figure 15.8.

Figure 15.8: Order of largest-known mixed circulant graphs of directed degree 3 and undirected degree 2, up to diameter 17: least-squares residual divided by the diameter cubed



For directed degree 2 and undirected degree 4, extremal and largest-known graphs have been discovered up to diameter 15 (see Appendix G). According to the tentative hypothesis shown in Table 15.8, the quartic coefficient of the fitted order polynomial would be $72/960$ or $3/40$. The value obtained by best fit up to diameter 15 is 0.08023. The closest simple fraction with denominator 960 is $77/960$, lying close to the

Figure 15.9: Order of largest-known mixed circulant graphs of directed degree 2 and undirected degree 4, up to diameter 15: least-squares residual divided by the diameter cubed

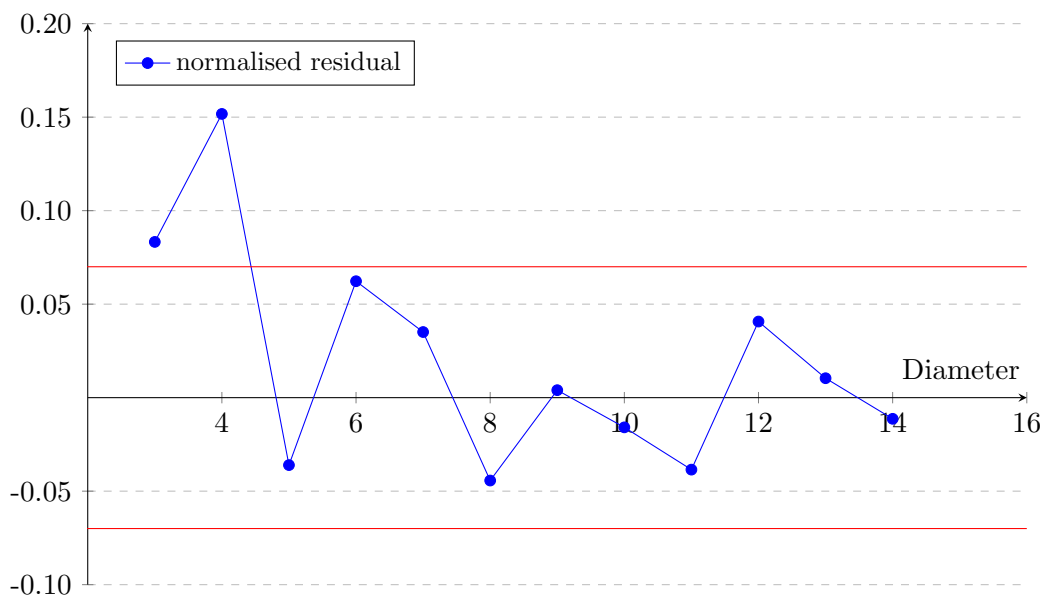


conjectured value of $72/960$. With an extra total squared residual of 0.3%, a quartic may be fitted to the graph orders for diameters up to 15 with leading coefficient $3/40$: $\hat{n}(k) = (3/40)k^4 + 0.4441k^3 + 4.498k^2 - 4.336k + 5.052$. Above a threshold diameter of 6, the normalised residuals vary within a range of ± 0.04 , see Figure 15.9.

For directed degree 1 and undirected degree 6, extremal and largest-known graphs have been discovered up to diameter 14 (see Appendix G). The tentative hypothesis for the quartic coefficient of the fitted order polynomial, shown in Table 15.8, is $88/480$ or $11/60$. The value obtained by best fit up to diameter 14 is 0.08748. The closest simple fraction with denominator 480 is $42/480$. This is only half the conjectured value. It is expected that higher diameters are required to stabilise the fit in order to validate the conjecture. Nevertheless, accepting an extra total squared residual of 21%, a quartic may be fitted with leading coefficient $11/60$:

$\hat{n} = (11/60)k^4 + 0.5576k^3 + 7.366k^2 - 26.73k + 51.76$. Above a threshold diameter of 4, the normalised residuals vary within a range of ± 0.07 , see Figure 15.10.

Figure 15.10: Order of largest-known mixed circulant graphs of directed degree 1 and undirected degree 6, up to diameter 14: least-squares residual divided by the diameter cubed



15.6 Extremal circulant graphs – a holistic perspective

In Section 2.5 we discussed the upper bound for the order of Abelian Cayley graphs of any directed and undirected degree established by López, Pérez-Rosés and Pujolàs [30], and determined the leading coefficient of the implied polynomial. For circulant graphs, this coefficient is $2^{\lfloor (d+1)/2 \rfloor} / f!$, where f is the dimension and d is the undirected degree. An aspirational goal is to discover an equivalent expression for extremal circulant graphs, covering directed, undirected and mixed graphs in all

degree combinations. The Extremal Order Conjecture 3.1 covers undirected graphs of arbitrary undirected degree. Conjecture 15.10 covers directed and mixed graphs up to undirected degree 3 of arbitrary directed degree, equally valid for the regular families of dimensions 1 and 2 and for the chaotic graph sets of dimension 3 and above. The dimension 4 results lend support to Conjecture 15.10. But results for higher dimensions are required in order to establish an evident pattern that can be represented by a common expression. Alternatively, perhaps there is some graph-theoretic approach that could enable such an expression to be derived analytically.

An open question is: what happens for directed and mixed circulant graphs of dimension 3 or more that causes the regular sequence of extremal orders to collapse into apparent chaos? One avenue that might yield the answer is a possible connection with chaos theory. There is a seminal paper on chaos theory by Li and Yorke from 1975 entitled *Period three implies chaos* [29]. They make the case that even very simple models with a single variable, such as the generalised logistic equation $x_{n+1} = rx_n(1 - x_n/K)$, can describe irregularities and chaotic oscillations of complicated phenomena. A special case of their main result says that if there is a periodic point in \mathbb{R} with period 3, then for each integer $n = 1, 2, 3, \dots$, there is a periodic point of period n and an uncountable number of points that are not even asymptotically periodic. In their analysis, a point a has period 3 under an endomorphism F if $a < F(a) < F^2(a)$ and $F^3(a) \leq a$ (or the reverse ordering). Perhaps there is some way in which this property or a similar one is shared by the extremal directed and mixed circulant graphs of dimension 3 and above. And if so, perhaps such an insight might be crucial in developing an analytical determination of the polynomials (or at least their leading coefficients) representing the trend curves fitted to the chaotic sequences, with increasing diameter, of extremal graph orders for directed and mixed circulant graphs, and in this way achieve the ultimate goal of a common expression for all categories of extremal Abelian Cayley graphs.

CONCLUSION

The focus of this thesis has been the degree-diameter problem for Abelian Cayley graphs, with emphasis on circulant graphs and on graphs that are members of infinite families. Prior to this work, largest-known graph families conjectured to be extremal had only been discovered up to degree 7, and there were no conjectures about how close to the legacy Abelian Cayley upper bound extremal graphs of any higher degree would be found. This thesis presents conjectures for the order of extremal graphs of any degree and arbitrary diameter for graphs in the following categories: circulant, Abelian Cayley, bipartite circulant, directed circulant and certain mixed circulant, and also for diameter 2 circulant of arbitrary degree. In all cases the conjectures are supported by largest-known graphs discovered by the author.

A most useful discovery was the format and properties of the canonical lattice generator matrix associated with an Abelian Cayley graph family. It was entirely unexpected to find that the LGM of an odd-degree quasimaximal family has off-diagonal elements that are always antisymmetric with magnitude 0 or 1 (eccentricity 0). The restriction that this property imposes on candidate LGMs was crucial in enabling successful computer searches for higher-degree graph families.

We now briefly review each chapter and indicate potential avenues for further research.

Chapter 2 summarised the published work in this field, including extremal and largest-known undirected and directed circulant graph families and the established lower and upper bounds on extremal graph order.

In Chapter 3, the original version of the Extremal Order Conjecture for circulant and Abelian Cayley graphs was presented. It was developed as a consequence of the degree 8 and 9 discoveries and their extension of the range of known results. This established the definition of a *quasimaximal* graph family, which was a pivotal concept in refining the search at higher degrees. More recent developments were reported in Chapter 13.

Chapter 4 presented largest-known circulant graph families up to degree 11 that were discovered using more traditional search methods for individual graphs, without the benefit of harnessing the structure of canonical quasimaximal LGMs. However, it was

by analysis of these results that the structure and properties of such LGMs were discovered.

Chapter 5 explored the number of maximal distance levels and the distance partition profile by vertex type for quasimaximal circulant graphs. As expected, the maximum levels are equivalent to non-overlapping Lee spheres in the corresponding lattice space with radius less than the graph diameter. The relation was also explored between quasimaximal defect and odd-girth defect, which is also explained by the associated LGM. Two important relationships between graph families were also discussed: conjugation and translation. We saw that conjugation is a property only of Abelian Cayley graph families that are quasimaximal. It would be interesting to research what property of quasimaximal families ensures that they alone admit conjugates.

Chapter 6 is probably the most important chapter for mathematical structures and relationships. The format and properties of canonical LGMs are fundamental to an understanding of quasimaximal graph families. A most useful consequence was the establishment of the Existence Proof Theorem for Abelian Cayley graph families. The equivalence of quasimaximal graph families, odd-girth-maximum families and radius-maximal LGMs and their implication that the LGM has eccentricity 0 is a powerful, unifying result. It would be an interesting research project to explore what equivalent structures and relationships might exist for directed Abelian Cayley graphs families, and even possibly the mixed case.

Chapter 7 explored the correspondence between all quasimaximal degree 7 circulant graph families of a diameter class and a category of canonical LGMs. It emerged that there exists a bijection between these two that enables a complete classification of all such graph families (infinitely many) and also explains precisely why graph families with some parameter sets exist and why others do not. This is conjectured to be equally true for the other diameter classes and for degree 6. We know that for higher degree, not all matrices in canonical quasimaximal LGM format have associated graph families. It might be an interesting research project to attempt to classify which of these matrices do represent families for degree 8 and above, and whether it is possible to devise a simple test for such matrices.

Chapter 8 presented the results of the optimised search methodology using candidate matrices in canonical quasimaximal LGM format to find complete graph families rather than individual graphs. Even so, by degree 20 the searches were limited by available computing power. Therefore, it is quite possible that some of these higher-degree largest-known circulant graph families are not extremal. A relatively

straightforward research project, given access to more powerful computers, would be to make exhaustive searches of all admissible matrices.

Chapter 9 established some relationships between quasimaximal circulant graph families and bipartite circulant graph families, and used these to generate largest-known bipartite circulant graph families up to degree 11. The same approach applied to the results of Chapter 8 would easily extend the range of largest-known bipartites up to degree 20.

Chapter 10 presented the results of the optimised search method enhanced to find Abelian Cayley graph families of any cyclic rank. Because of the extra complexity induced by cyclic rank, these results only run to degree 15 and quite likely are not extremal. Extending the scope to higher degree would be another potential research project, given greater computing power. But such a search would not necessarily reveal any interesting new mathematics.

Chapter 11 was a surprising diversion from the main flow of the thesis. The discovery that Lucas polynomials could be used to construct order polynomials and generating set polynomials for circulant graphs was serendipity. Moreover, they form an infinite set of quasimaximal families, they are arc-transitive, and their associated LGMs have a very simple format. The related research potential could be to investigate what other LGM formats generate arc-transitive graph families, and whether similar sets to these Lucas circulant graph families exist, perhaps for more diameter classes and with higher order.

Chapter 12 presented two sequences of LGMs conjectured to generate circulant and Abelian Cayley graph families of any dimension, similar to the Lucas circulant graph sequence but with higher order. The families have order polynomials with third coefficient that increases linearly with dimension. An interesting research project would be to find a sequence of LGMs where the third coefficient increased with the square of the dimension, in line with the revised Extremal Order Conjecture in the following chapter.

Chapter 13 extended the Extremal Order Conjecture to the third coefficient. Noting that with increasing degree the diameter threshold, below which extremal graphs are not members of extremal families, also increases, the question arises whether all graphs are members of families or if some graphs are sporadic. If some are sporadic, then finding extremal families would not guarantee that extremal graph had been found. A limited study of potentially sporadic extremal graphs found families of

which they were members in all cases, supporting the conjecture that all graphs belong to families.

Section 13.4 on higher asymptotic lower bounds probably offers the greatest opportunity for significant advance in this field. Rogers' theorem on the existence of lattice coverings with asymptotically low density contradicts the Extremal Order Conjecture 3.1. However, there are doubts about its validity and the conjecture is considered to remain valid. It would be very satisfying to prove the matter one way or the other.

Chapter 14 is an exception in exploring circulant and Abelian Cayley graphs of fixed diameter 2 for arbitrary degree. It remains an open research question, for circulant graphs, whether the leading coefficient of $3/8$ is an upper bound, as well as the more general conjectured upper bound of $(k+1)/(2(k+2)^{k-1})$ for any diameter k .

Chapter 15 covered directed and mixed circulant graphs. The main finding is that whereas extremal and largest-known directed and mixed circulant graphs of dimensions 1 and 2 are members of families with order and generating sets defined by polynomials in the diameter, this does not hold for dimension 3 and is conjectured also not to hold for all higher dimensions. Instead, the series appears to be chaotic, with no discernable pattern, although the order does approach a polynomial in the diameter asymptotically for each directed/undirected degree combination studied. This is another area with rich research possibilities. Important questions include:

- Why is the order of directed and mixed circulant graphs of dimension 3 and above chaotic for increasing diameter?
- Despite being chaotic, is it true that their order asymptotically approaches a polynomial in the diameter of degree equal to the dimension?
- If so, is there a common formula for the leading coefficient of this polynomial across all directed/undirected degree combinations, as there is for the best upper bound?

BIBLIOGRAPHY

- [1] D. Bevan, G. Erskine and R. Lewis, *Large circulant graphs of fixed diameter and arbitrary degree*, Ars Mathematica Contemporanea, **13** (2017) no. 2, 275-291.
- [2] F. Boesch and J-F Wang, *Reliable circulant networks with minimum transmission delay*, IEEE Trans. Circuits and Systems, **32** (12) (1985) 1286-1291.
- [3] S. Chen and X.D. Jia, *Undirected loop networks*, Networks, **23** (1993) 257-260.
- [4] J.H. Conway, R.A. Parker and N.J.A. Sloane, *The covering radius of the Leech lattice*, Proceedings of the Royal Society, London **A380** (1982) 261-290.
- [5] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, New York, Springer (1999).
- [6] J. Cullinan and F. Hajir, *Primes of prescribed congruence class in short intervals*, Integers **12** (2012): A56.
- [7] *Combinatorics Wiki - The Degree-Diameter Problem*, [combinatoricswiki:The Degree-Diameter Problem](#).
- [8] C. Dalfó, M.A. Fiol and N. López, *New results on the degree/diameter problem of mixed Abelian Cayley graphs*, [arXiv:1908.00245](#) [math.CO] (2019).
- [9] C. Dalfó, M.A. Fiol, N. López and J. Ryan, *An improved Moore bound and some new optimal families of mixed Abelian Cayley graphs*, [arXiv:2006.02897](#) [math.CO] (2020).
- [10] R. Dougherty and V. Faber, *The degree-diameter problem for several varieties of Cayley graphs, I: The Abelian case*, SIAM Journal on Discrete Mathematics, **17** (2004) no. 3, 478-519.
- [11] R. Dougherty and V. Faber, *The degree-diameter problem for several varieties of Cayley graphs, I: The Abelian case*, Los Alamos National Laboratory E-print server, <http://www.c3.lanl.gov/dm/pub/laces.html> (1994)
- [12] *Encyclopedia of Mathematics - Lucas polynomials*, [Encyclopedia of Mathematics:Lucas polynomials](#).
- [13] R. Fera-Purón, J. Ryan and H. Pérez-Rosés, *Searching for large multi-loop networks*, Electronic Notes in Discrete Mathematics **46** (2014) 233-240.
- [14] C.M. Fiduccia, R.W. Forcade and J.S. Zito, *Geometry and diameter bounds of directed Cayley graphs of Abelian groups*, SIAM Journal on Discrete Mathematics, **11** (1998) no. 1 157-167.
- [15] C. Garcia and C. Peyrat, *Large Cayley graphs on an Abelian group*, Discrete Applied Mathematics, **75** (1997) 125-133.
- [16] C. Godsil and G. Royle, *Algebraic Graph Theory*, New York, Springer-Verlag (2001).
- [17] S.W. Golomb and L.R. Welch, *Perfect codes in the Lee metric and the packing of polyominoes*, SIAM Journal on Applied Mathematics, **18**(2) (1970) 302-317.
- [18] T.S. Griggs, personal communication, (May 2016).
- [19] P. Gritzmann, *Lattice covering of space with symmetric convex bodies*, Mathematika, **32** (1985) 311-315.
- [20] A.J. Hoffman and R.R. Singleton, *On Moore graphs with diameters 2 and 3*, IBM Journal of research and development, **5**(4) (1960) 497-504.
- [21] P. Horak, *Tilings in Lee metric*, European Journal of Combinatorics, **30** (2009) 480-489.
- [22] D.F. Hsu and X.D. Jia, *Extremal problems in the construction of distributed loop networks*, SIAM Journal on Discrete Mathematics, **7** (1994) no. 1 57-71.

- [23] R.R. Lewis, *The degree-diameter problem for circulant graphs of degree 8 and 9*, Electronic Journal of Combinatorics, **21(4)** (2014) #P4.50.
- [24] R.R. Lewis, *Distance partitions of extremal and largest known circulant graphs of degree 2 to 9*, [arXiv:1408.0988](#) [math.CO] (2014).
- [25] R.R. Lewis, *Improved upper bounds for the order of some classes of Abelian Cayley and circulant graphs of diameter two*, Journal of Interconnection Networks, **17(3 & 4)** (2017) 1741012 (15 pages).
- [26] R.R. Lewis, *The degree-diameter problem for circulant graphs of degree 10 and 11*, Discrete Mathematics, **341(9)** (2018) 2553-2566.
- [27] R.R. Lewis, *Full proof of the existence of a degree 8 circulant graph of order $L(8, k)$ of arbitrary diameter k* , [arXiv:1404.3949](#) [math.CO] (2014).
- [28] R.R. Lewis, *The degree-diameter problem for circulant graphs of degrees 10 and 11 - extended version*, [arXiv:1803.07071](#) [math.CO] (2018).
- [29] T. Y. Li and J. A. Yorke, *Period three implies chaos*, The American Mathematical Monthly, **82** (1975) no. 10, 985-992.
- [30] N. López, H. Pérez-Rosés and J. Pujolàs, *The degree/diameter problem for mixed Abelian Cayley graphs*, Discrete Applied Mathematics, **231** (2017) 190-197.
- [31] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, American Journal of Mathematics, **1(2)** (1878) 184-240, 299-321.
- [32] H. Macbeth, J. Šiagiová and J. Širáň, *Cayley graphs of given degree and diameter for cyclic, Abelian and metacyclic groups*, Discrete Mathematics, **312** (2012) 94-99.
- [33] B. McKay, *Circulant graphs of diameter 2*, personal communication to J. Širáň (2004).
- [34] B. McKay, M. Miller and J. Širáň, *A note on large graphs of diameter two and given maximum degree*, Journal of Combinatorial Theory, Series B **74** (1) (1998) 110-118.
- [35] M. Miller and J. Širáň, *Moore graphs and beyond: A survey of the degree/diameter problem*, The Electronic Journal of Combinatorics, Dynamic survey, **DS14v2** (2013) 92pp.
- [36] E. Monakhova, *Synthesis of optimal Diophantine structures*, Computer Systems Novosibirsk, **80** (1979) 18-35.
- [37] E. Monakhova, *Optimal Triple Loop Networks with Given Transmission Delay: Topological Design and Routing*, Inter. Network Optimization Conference (INOC '2003), Evry/Paris, France, (2003) 410-415.
- [38] E. Monakhova, *On a construction of quadruple circulant networks with the maximal number of nodes for any diameter*, Prikladnaya Diskretnaya Matematika, no. 3, (2013) 76-85.
- [39] E. Monakhova, *On a construction of multidimensional circulant graphs of diameter two*, Izvestiya Tomskogo Politehnicheskogo Universiteta **323** 2 (2013) 25-28.
- [40] F.P. Muga, *Undirected circulant graphs*, Inter. Symp. on Parallel Architectures, Algorithms and Networks. IEEE, (1994) 113-118.
- [41] V.K. Nguyen, *Family of circulant graphs and its expander properties*, San Jose State University masters thesis, (2010).
- [42] OEIS Foundation Inc. (2020), *The On-Line Encyclopedia of Integer Sequences*, [OEIS:A162514](#).
- [43] O. Ordentlich, O. Regev and B. Weiss, *New bounds on the density of lattice coverings*, [arXiv:2006.00340v1](#) [math.NT] (2020).
- [44] A. Pott and Y. Zhou, *Cayley graphs of diameter two from difference sets*, Journal of graph theory **85(2)** (2017) 533-544.
- [45] O. Ramaré and R. Rumely, *Primes in arithmetic progressions*, Mathematics of Computation, **65(213)** (1996) 397-425.
- [46] C.A. Rogers, *Lattice coverings of space*, Mathematika, **6** (1959) 33-39.

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- [47] C.A. Rogers, *Packing and Covering*, Cambridge, UK, Cambridge University Press (1964).
 - [48] R.G. Stanton and D.D. Cowan, *Note on a square functional equation*, SIAM Review, **12(2)** (1970) 277-279.
 - [49] D. Tzvieli, *Minimal diameter double-loop networks. 1. Large infinite optimal families*, Networks, **21** (1991) 387-415.
 - [50] T. Vetrík, *Abelian Cayley graphs of given degree and diameter 2 and 3*, Graphs and Combinatorics, **30(6)** (2014), 1587-1591.
 - [51] C.K. Wong and D. Coppersmith, *A combinatorial problem related to multimode memory organizations*, J. Assoc Comput Mach. **21** (1974) 392-402.
 - [52] J. Yebra, M. Fiol, P. Morillo and I Alegre, *The diameter of undirected graphs associated to plane tessellations*, Ars Combin., **20B** (1985) 159-171.

APPENDIX A

EXTREMAL AND LARGEST-KNOWN CIRCULANT GRAPH FAMILIES

This appendix documents extremal and largest-known circulant graph families up to degree 20. As a reminder, according to Definition 1.4, a graph family is an infinite set of graphs of given degree d and dimension $f = \lfloor d/2 \rfloor$, defined for each diameter k of a diameter class, with order and generating set specified by polynomials in k of maximum degree f . Unless otherwise stated, the diameter class is modulo f for odd dimension and modulo $f/2$ for even. The graph families are identified by a code, such as F6:0B. In this example, F indicates that it is a circulant graph family, 6 is the degree, 0 is the diameter class (mod 3), and B is the isomorphism class (where there is more than one).

Of the largest-known circulant graph families presented in this appendix, the following have been discovered by the author:

- Degree 6 - F6:0B and F6:2B
- Degree 7 - F7:1B and F7:2B
- Degree 8 and above - all families

All these extremal and largest-known circulant graph families are quasimaximal. Above some low diameter thresholds, the graphs are extremal or largest known and have maximum odd girth. These thresholds are given in the tables. The graphs up to diameter 16 are included in Appendix D, with reference to their families by isomorphism class. In addition, for each graph family, the number of maximal levels in the distance partition of each graph is a linear function of the diameter: $2k/f$ plus a small constant, for diameter k and dimension f . This is also indicated in the tables.

A.1 Circulant graph families of degrees 4 and 5

Table A.1: Degree 4, all diameters k , $a = k$

LGM odd basis	LGM	Polynomial in $2a$		
Family F4 (self-transpose, self-conjugate). Graphs are extremal from $k = 1$. odd-girth maximum from $k = 1$. Maximal levels: k from $k = 1$.				
$\begin{pmatrix} 2a+1 & -1 \\ 1 & 2a+1 \end{pmatrix}$	$\begin{pmatrix} a & -a-1 \\ a+1 & a \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	/ 2
		g_1		
		g_2		

Table A.2: Degree 5, all diameters k , $a = k$

LGM	Polynomial in $2a$		
Family F5 (self-transpose, self-conjugate). Graphs are extremal from $k = 1$. odd-girth maximum from $k = 1$. Maximal levels: $k - 1$ from $k = 1$.			
$\begin{pmatrix} 2a+1 & -1 \\ 1 & 2a-1 \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	
	g_1		
	g_2		

A.2 Circulant graph families of degrees 6 and 7

Table A.3: Degree 6, diameter class $k \equiv 0 \pmod{3}$, $a = 2k/3$

LGM odd basis	LGM	Polynomial in $2a$				
Family F6:0A (self-transpose, conjugate of F6:2A, translate of F7:2A). Graphs are largest known from $k = 3$. odd-girth maximum from $k = 3$. Maximal levels: $2k/3$ from $k = 3$.						
$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$	$\begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a & -a & a+1 \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$	/	2	
		g_1	$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$			
		g_2	$\begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$			
		g_3	$\begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix}$			
Family F6:0B (self-transpose, conjugate of F6:2B, translate of F7:2B). Graphs are largest known from $k = 3$. odd-girth maximum from $k = 3$. Maximal levels: $2k/3$ from $k = 3$.						
$\begin{pmatrix} 2a+2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	$\begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a+1 & -a & a \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	/	2	
		g_1	$\begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}$			
		g_2	$\begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$			
		g_3	$\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$			

Table A.4: Degree 6, diameter class $k \equiv 1 \pmod{3}$, $a = (2k+1)/3$

LGM odd basis	LGM	Polynomial in $2a$				
Family F6:1 (self-transpose, self-conjugate, translate of F7:0). Graphs are largest known from $k = 1$. odd-girth maximum from $k = 1$. Maximal levels: $(2k + 1)/3$ from $k = 1$.						
$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	$\begin{pmatrix} a - 1 & -a - 1 & -a \\ a & a - 1 & -a - 1 \\ a & -a & a \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	/	2	
		g_1				
		g_2				
		g_3				

Table A.5: Degree 6, diameter class $k \equiv 2 \pmod{3}$, $a = (2k + 2)/3$

LGM odd basis	LGM	Polynomial in $2a$
Family F6:2A (self-transpose, conjugate of F6:0A, translate of F7:1A). Graphs are largest known from $k = 2$. odd-girth maximum from $k = 2$. Maximal levels: $(2k - 1)/3$ from $k = 2$.		
$\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	$\begin{pmatrix} a-1 & -a-1 & -a+1 \\ a & a-1 & -a \\ a-1 & -a & a \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 3 & -2 \end{pmatrix} / 2$ $g_1 \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ $g_2 \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}$ $g_3 \begin{pmatrix} 0 & 1 & -1 & 1 \end{pmatrix}$
Family F6:2B (self-transpose, conjugate of F6:0B, translate of F7:1B). Graphs are largest known from $k = 2$. odd-girth maximum from $k = 2$. Maximal levels: $(2k - 1)/3$ from $k = 2$.		
$\begin{pmatrix} 2a-2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	$\begin{pmatrix} a-2 & -a-1 & -a \\ a-1 & a-1 & -a-1 \\ a-1 & -a & a \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 3 & -2 \end{pmatrix} / 2$ $g_1 \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}$ $g_2 \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$ $g_3 \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$

Table A.6: Degree 7, diameter class $k \equiv 0 \pmod{3}$, $a = 2k/3$

LGM	Polynomial in $2a$
Family F7:0 (self-transpose, self-conjugate, translate of F6:1). Graphs are largest known from $k = 3$. odd-girth maximum from $k = 3$. Maximal levels: $2k/3$ from $k = 3$.	
$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	Order $\begin{pmatrix} 1 & 0 & 3 & 0 \end{pmatrix}$ $g_1 \begin{pmatrix} 0 & 1 & 2 & 1 \end{pmatrix}$ $g_2 \begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}$ $g_3 \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}$

Table A.7: Degree 7, diameter class $k \equiv 1 \pmod{3}$, $a = (2k + 1)/3$

LGM	Polynomial in $2a$
Family F7:1A (self-transpose, conjugate of F7:2A, translate of F6:2A). Graphs are largest known from $k = 4$. odd-girth maximum from $k = 1$. Maximal levels: $(2k - 2)/3$ from $k = 1$.	
$\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 3 & -2 \end{pmatrix}$ $g_1 \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$ $g_2 \begin{pmatrix} 0 & 1 & -2 & 1 \end{pmatrix}$ $g_3 \begin{pmatrix} 0 & 1 & -2 & 3 \end{pmatrix}$
Family F7:1B (self-transpose, conjugate of F7:2B, translate of F6:2B). Graphs are largest known from $k = 4$. odd-girth maximum from $k = 1$. Maximal levels: $(2k - 2)/3$ from $k = 1$.	
$\begin{pmatrix} 2a-2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 3 & -2 \end{pmatrix}$ $g_1 \begin{pmatrix} 0 & 1 & 2 & 1 \end{pmatrix}$ $g_2 \begin{pmatrix} 0 & 1 & -2 & -3 \end{pmatrix}$ $g_3 \begin{pmatrix} 0 & 1 & -4 & 3 \end{pmatrix}$

Table A.8: Degree 7, diameter class $k \equiv 2 \pmod{3}$, $a = (2k - 1)/3$

LGM	Polynomial in $2a$
Family F7:2A (self-transpose, conjugate of F7:1A, translate of F6:0A). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 2$. Maximal levels: $(2k - 1)/3$ from $k = 2$.	
$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$	Order $\begin{pmatrix} 1 & 2 & 3 & 2 \\ g_1 & (0 & 1 & 2 & 3) \\ g_2 & (0 & 1 & 2 & 1) \\ g_3 & (0 & 1 & 0 & 1) \end{pmatrix}$
Family F7:2B (self-transpose, conjugate of F7:1B, translate of F6:0B). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 2$. Maximal levels: $(2k - 1)/3$ from $k = 2$.	
$\begin{pmatrix} 2a+2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	Order $\begin{pmatrix} 1 & 2 & 3 & 2 \\ g_1 & (0 & 1 & 0 & 1) \\ g_2 & (0 & 0 & 1 & 1) \\ g_3 & (0 & 0 & 1 & -1) \end{pmatrix}$

A.3 Circulant graph families of degrees 8 and 9

Table A.9: Degree 8, diameter class $k \equiv 0 \pmod{2}$, $a = k/2$

LGM odd basis	Polynomial in $2a$
Family F8:0 (self-transpose, conjugate of F8:1). Graphs are largest known from $k = 4$. odd-girth maximum from $k = 2$. Maximal levels: $k/2$ from $k = 2$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & -1 \\ 1 & 1 & 1 & 2a+1 \end{pmatrix}$	Order $\begin{pmatrix} 1 & 2 & 6 & 4 & 0) & / & 2 \\ g_1 & (0 & 0 & 0 & 4 & 0) & / & 2 \\ g_2 & (0 & 0 & 1 & 0 & -2) & / & 2 \\ g_3 & (0 & 1 & 0 & 2 & 0) & / & 2 \\ g_4 & (0 & 1 & 3 & 6 & 2) & / & 2 \end{pmatrix}$

Table A.10: Degree 8, diameter class $k \equiv 1 \pmod{2}$, $a = (k + 1)/2$

LGM odd basis	Polynomial in $2a$
Family F8:1 (self-transpose, conjugate of F8:0). Graphs are largest known from $k = 3$. odd-girth maximum from $k = 3$. Maximal levels: $(k + 1)/2$ from $k = 3$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & -1 \\ 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 6 & -4 & 0) & / & 2 \\ g_1 & (0 & 0 & 0 & 2 & 0) & / & 2 \\ g_2 & (0 & 0 & 1 & -2 & 2) & / & 2 \\ g_3 & (0 & 1 & -2 & 4 & 0) & / & 2 \\ g_4 & (0 & 1 & -1 & 4 & -2) & / & 2 \end{pmatrix}$

Table A.11: Degree 9, diameter class $k \equiv 0 \pmod{2}$, $a = k/2$

LGM	Polynomial in $2a$
Family F9:0 (self-transpose). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 2$. Maximal levels: $k/2$ from $k = 2$.	
$\begin{pmatrix} 2a+1 & 0 & 0 & -1 \\ 0 & 2a & -1 & -1 \\ 0 & 1 & 2a & -1 \\ 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order $(1 \ 0 \ 3 \ 2 \ 0)$ $g_1 \ (0 \ 1 \ 0 \ 2 \ 1)$ $g_2 \ (0 \ 1 \ 1 \ 2 \ 1)$ $g_3 \ (0 \ 0 \ 0 \ 2 \ 1)$ $g_4 \ (0 \ 1 \ -1 \ 1 \ 1)$

Table A.12: Degree 9, diameter class $k \equiv 1 \pmod{2}$, $a = (k+1)/2$

LGM	Polynomial in $2a$
Family F9:1a (transpose of F9:1b). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 3$. Maximal levels: $(k-1)/2$ from $k = 3$.	
$\begin{pmatrix} 2a-2 & 1 & 0 & 1 \\ -1 & 2a-1 & 1 & 1 \\ 0 & -1 & 2a-1 & 0 \\ -1 & -1 & 0 & 2a \end{pmatrix}$	Order $(1 \ -4 \ 9 \ -10 \ 4)$ $g_1 \ (0 \ 1 \ -5 \ 10 \ -7)$ $g_2 \ (0 \ 1 \ -3 \ 1 \ 1)$ $g_3 \ (0 \ 1 \ -2 \ 4 \ -5)$ $g_4 \ (0 \ 0 \ 2 \ -4 \ 3)$
Family F9:1b (transpose of F9:1a). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 3$. Maximal levels: $(k-1)/2$ from $k = 3$.	
$\begin{pmatrix} 2a-2 & -1 & 0 & -1 \\ 1 & 2a-1 & -1 & -1 \\ 0 & 1 & 2a-1 & 0 \\ 1 & 1 & 0 & 2a \end{pmatrix}$	Order $(1 \ -4 \ 9 \ -10 \ 4)$ $g_1 \ (0 \ 1 \ -3 \ 7 \ -7)$ $g_2 \ (0 \ 0 \ 1 \ -2 \ 1)$ $g_3 \ (0 \ 1 \ -3 \ 5 \ -3)$ $g_4 \ (0 \ 1 \ -3 \ 5 \ -5)$

A.4 Circulant graph families of degrees 10 and 11

Table A.13: Degree 10, diameter class $k \equiv 0 \pmod{5}$, $a = 2k/5$

LGM odd basis	Polynomial in $2a$
Family F10:0 (self-transpose, conjugate of F10:4, translate of F11:3). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 5$. Maximal levels: $2k/5$ from $k = 5$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & 0 & -1 \\ 1 & 1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order $(1 \ 2 \ 8 \ 8 \ 5 \ 2) / 2$ $g_1 \ (0 \ 1 \ 1 \ 2 \ 0 \ 0) / 2$ $g_2 \ (0 \ 0 \ 2 \ 3 \ 3 \ 2) / 2$ $g_3 \ (0 \ 0 \ 2 \ 0 \ 0 \ 0) / 2$ $g_4 \ (0 \ 0 \ 1 \ 3 \ 2 \ 0) / 2$ $g_5 \ (0 \ 1 \ 2 \ 4 \ 5 \ 2) / 2$

Table A.14: Degree 10, diameter class $k \equiv 1 \pmod{5}$, $a = (2k + 3)/5$

LGM odd basis	Polynomial in $2a$
Family F10:1 (self-transpose, conjugate of F10:3). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 6$. Maximal levels: $(2k + 3)/5$ from $k = 6$.	
$\begin{pmatrix} 2a & -1 & -1 & 0 & 0 \\ 1 & 2a-1 & 0 & -1 & -1 \\ 1 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order (1 -4 13 -20 14 -4) / 2
	g_1 (0 2 -7 25 -38 18) / 2
	g_2 (0 0 1 -4 0 2) / 2
	g_3 (0 1 -2 6 -10 6) / 2
	g_4 (0 2 -6 15 -13 2) / 2
	g_5 (0 1 -6 14 -23 14) / 2

Table A.15: Degree 10, diameter class $k \equiv 2 \pmod{5}$, $a = (2k + 1)/5$

LGM odd basis	Polynomial in $2a$
Family F10:2 (self-transpose, self-conjugate, translate of F11:0). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 2$. Maximal levels: $(2k + 1)/5$ from $k = 2$.	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a & 0 & 0 \\ 1 & 0 & 0 & 2a & -1 \\ 1 & 0 & 0 & 1 & 2a \end{pmatrix}$	Order (1 0 6 0 5 0) / 2
	g_1 (0 0 0 2 0 2) / 2
	g_2 (0 1 0 5 2 2) / 2
	g_3 (0 0 1 0 3 2) / 2
	g_4 (0 0 1 0 3 -2) / 2
	g_5 (0 1 0 5 -2 2) / 2

Table A.16: Degree 10, diameter class $k \equiv 3 \pmod{5}$, $a = (2k - 1)/5$

LGM odd basis	Polynomial in $2a$
Family F10:3 (self-transpose, conjugate of F10:1). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 3$. Maximal levels: $(2k - 1)/5$ from $k = 3$.	
$\begin{pmatrix} 2a & -1 & -1 & 0 & 0 \\ 1 & 2a+1 & 0 & -1 & -1 \\ 1 & 0 & 2a+1 & -1 & -1 \\ 0 & 1 & 1 & 2a+1 & -1 \\ 0 & 1 & 1 & 1 & 2a+1 \end{pmatrix}$	Order (1 4 13 20 14 4) / 2
	g_1 (0 2 11 33 46 22) / 2
	g_2 (0 1 2 -2 -8 -6) / 2
	g_3 (0 2 5 8 2 -2) / 2
	g_4 (0 0 2 5 -7 -10) / 2
	g_5 (0 1 0 8 11 2) / 2

Table A.17: Degree 10, diameter class $k \equiv 4 \pmod{5}$, $a = (2k + 2)/5$

LGM odd basis	Polynomial in $2a$
Family F10:4 (self-transpose, conjugate of F10:0, translate of F11:2). Graphs are largest known from $k = 4$. odd-girth maximum from $k = 4$. Maximal levels: $(2k + 2)/5$ from $k = 4$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & 0 \\ 1 & 2a-1 & -1 & 0 & -1 \\ 1 & 1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order (1 -2 8 -8 5 -2) / 2
	g_1 (0 0 0 2 -2 1)
	g_2 (0 0 1 -3 1 -1)
	g_3 (0 0 1 0 1 -1)
	g_4 (0 0 1 1 0 1)
	g_5 (0 1 -1 4 -2 1)

Table A.18: Degree 11, diameter class $k \equiv 0 \pmod{5}$, $a = 2k/5$

LGM	Polynomial in $2a$						
Family F11:0 (self-transpose, self-conjugate, translate of F10:2). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 5$. Maximal levels: $2k/5$ from $k = 5$.							
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a & 0 & 0 \\ 1 & 0 & 0 & 2a & -1 \\ 1 & 0 & 0 & 1 & 2a \end{pmatrix}$	Order	(1	0	6	0	5	0)
	g_1	(0	1	0	2	0	1)
	g_2	(0	1	0	4	4	-1)
	g_3	(0	1	-2	6	-6	1)
	g_4	(0	1	2	6	6	1)
	g_5	(0	1	0	4	-4	-1)

Table A.19: Degree 11, diameter class $k \equiv 1 \pmod{5}$, $a = (2k - 2)/5$

LGM	Polynomial in $2a$						
Family F11:1a (transpose of F11:1b). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 6$. Maximal levels: $(2k - 2)/5$ from $k = 6$.							
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a+1 & 1 & 0 \\ 1 & 0 & -1 & 2a+1 & -1 \\ 1 & 0 & 0 & 1 & 2a+2 \end{pmatrix}$	Order	(1	4	12	20	15	4)
	g_1	(0	1	4	12	16	7)
	g_2	(0	1	4	10	14	3)
	g_3	(0	1	2	6	4	3)
	g_4	(0	1	4	12	22	9)
	g_5	(0	1	2	8	4	1)
Family F11:1b (transpose of F11:1a). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 6$. Maximal levels: $(2k - 2)/5$ from $k = 6$.							
$\begin{pmatrix} 2a & 1 & 1 & 1 & 1 \\ -1 & 2a & 1 & 0 & 0 \\ -1 & -1 & 2a+1 & -1 & 0 \\ -1 & 0 & 1 & 2a+1 & 1 \\ -1 & 0 & 0 & -1 & 2a+2 \end{pmatrix}$	Order	(1	4	12	20	15	4)
	g_1	(0	1	6	18	26	13)
	g_2	(0	1	4	8	12	5)
	g_3	(0	1	2	10	16	9)
	g_4	(0	1	2	4	4	3)
	g_5	(0	1	2	4	-2	-3)

Table A.20: Degree 11, diameter class $k \equiv 2 \pmod{5}$, $a = (2k + 1)/5$

LGM	Polynomial in $2a$						
Family F11:2 (self-transpose, conjugate of F11:3, translate of F10:4). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 2$. Maximal levels: $(2k + 1)/5$ from $k = 2$.							
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & 0 \\ 1 & 2a-1 & 1 & 0 & -1 \\ 1 & -1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order	(1	-2	8	-8	5	-2)
	g_1	(0	1	0	-2	2	-1)
	g_2	(0	1	-4	4	-2	1)
	g_3	(0	1	2	-2	2	-1)
	g_4	(0	1	-4	6	-4	1)
	g_5	(0	1	-2	2	-2	1)

Table A.21: Degree 11, diameter class $k \equiv 3 \pmod{5}$, $a = (2k - 1)/5$

LGM	Polynomial in $2a$
Family F11:3 (self-transpose, conjugate of F11:2, translate of F10:0). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 3$. Maximal levels: $(2k - 1)/5$ from $k = 3$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & 1 & 0 & -1 \\ 1 & -1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order (1 2 8 8 5 2) g_1 (0 1 4 8 6 3) g_2 (0 1 2 2 4 1) g_3 (0 1 0 0 0 1) g_4 (0 1 2 4 0 -1) g_5 (0 1 0 6 6 1)

Table A.22: Degree 11, diameter class $k \equiv 4 \pmod{5}$, $a = (2k + 2)/5$

LGM	Polynomial in $2a$
Family F11:4 (self-transpose). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 4$. Maximal levels: $(2k + 2)/5$ from $k = 4$.	
$\begin{pmatrix} 2a+1 & 0 & -1 & 0 & -1 \\ 0 & 2a+1 & 0 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 \\ 0 & 1 & 1 & 2a & 1 \\ 1 & 1 & 1 & -1 & 2a-1 \end{pmatrix}$	Order (1 -4 12 -16 9 -4) g_1 (0 1 -4 8 -12 5) g_2 (0 1 -4 8 -4 -3) g_3 (0 1 -4 8 -4 5) g_4 (0 1 -4 16 -12 5) g_5 (0 1 0 4 0 1)

A.5 Circulant graph families of degrees 12 and 13

Table A.23: Degree 12, diameter class $k \equiv 0 \pmod{3}$, $a = k/3$

LGM odd basis	Polynomial in $2a$
Family F12:0 (self-transpose). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 3$. Maximal levels: $k/3$ from $k = 3$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 2a+1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 2a & 1 & -1 & 0 \\ 1 & 0 & -1 & 2a & -1 & -1 \\ 0 & 1 & 1 & 1 & 2a & 1 \\ 0 & 0 & 0 & 1 & -1 & 2a \end{pmatrix}$	Order (1 2 11 14 13 6 0) /2 g_1 (0 2 4 24 16 13 0) /2 g_2 (0 0 1 -3 -5 -9 0) /2 g_3 (0 2 3 14 8 9 2) /2 g_4 (0 0 2 1 0 1 -2) /2 g_5 (0 1 -1 2 -6 -5 -2) /2 g_6 (0 1 3 8 15 7 0) /2

Table A.24: Degree 12, diameter class $k \equiv 1 \pmod{3}$, $a = (k-1)/3$

LGM odd basis						Polynomial in $2a$									
Family F12:1a (transpose of F12:1b). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 4$. Maximal levels: $(k + 2)/3$ from $k = 4$.															
$\begin{pmatrix} 2a + 2 & -1 & 0 & 0 & 0 & -1 \\ 1 & 2a + 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 2a + 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 2a + 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2a + 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 2a \end{pmatrix}$						Order	(1	6	24	58	75	46	10)	/2	
						g_1	(0	1	4	12	17	10	2)	/2	
						g_2	(0	0	2	10	20	16	4)	/2	
						g_3	(0	0	1	2	-3	-6	-2)	/2	
						g_4	(0	0	0	5	14	14	4)	/2	
						g_5	(0	1	5	14	25	22	6)	/2	
						g_6	(0	0	2	7	11	8	2)	/2	
Family F12:1b (transpose of F12:1a). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 4$. Maximal levels: $(k + 2)/3$ from $k = 4$.															
$\begin{pmatrix} 2a + 2 & 1 & 0 & 0 & 0 & 1 \\ -1 & 2a + 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 2a + 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 2a + 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 2a + 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 2a \end{pmatrix}$						Order	(1	6	24	58	75	46	10)	/2	
						g_1	(0	1	2	10	15	4	-2)	/2	
						g_2	(0	0	2	16	36	38	16)	/2	
						g_3	(0	0	1	10	39	48	18)	/2	
						g_4	(0	0	2	7	6	-4	-4)	/2	
						g_5	(0	1	7	26	45	32	6)	/2	
						g_6	(0	2	12	39	77	78	12)	/2	

Table A.25: Degree 12, diameter class $k \equiv 2 \pmod{3}$, $a = (k+1)/3$

LGM odd basis	Polynomial in $2a$								
Family F12:2a (transpose of F12:2b). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 2$. Maximal levels: $(k + 1)/3$ from $k = 2$.									
$\begin{pmatrix} 2a - 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 2a - 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 2a - 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 2a & 0 & 1 \\ 1 & 1 & 1 & 0 & 2a & 1 \\ 1 & 0 & 0 & -1 & -1 & 2a + 1 \end{pmatrix}$	Order	(1	-2	11	-12	-2	4	0)	/2
	g_1	(0	2	2	14	-16	-2	0)	/2
	g_2	(0	0	1	-13	-2	12	0)	/2
	g_3	(0	1	0	-3	-4	8	0)	/2
	g_4	(0	1	-5	7	5	-6	2)	/2
	g_5	(0	2	-7	18	-7	-8	2)	/2
	g_6	(0	0	1	-15	26	-12	0)	/2
Family F12:2b (transpose of F12:2a). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 2$. Maximal levels: $(k + 1)/3$ from $k = 2$.									
$\begin{pmatrix} 2a - 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 2a - 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 2a - 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 2a & 0 & -1 \\ -1 & -1 & -1 & 0 & 2a & -1 \\ -1 & 0 & 0 & 1 & 1 & 2a + 1 \end{pmatrix}$	Order	(1	-2	11	-12	-2	4	0)	/2
	g_1	(0	0	2	-2	0	0	0)	/2
	g_2	(0	0	1	3	0	-2	0)	/2
	g_3	(0	1	-2	7	-2	-2	0)	/2
	g_4	(0	0	1	-6	5	2	-2)	/2
	g_5	(0	1	-1	5	-7	0	2)	/2
	g_6	(0	0	1	1	-4	2	0)	/2

Table A.26: Degree 13, diameter class $k \equiv 0 \pmod{3}$, $a = k/3$

LGM	Polynomial in $2a$
Family F13:0a (transpose of F13:0b). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 3$. Maximal levels: $k/3$ from $k = 3$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 2a+1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 & -1 \\ 0 & 0 & 1 & 2a & 1 & 0 \\ 1 & 1 & 1 & -1 & 2a-1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$	Order (1 0 8 2 -1 -4 0) g_1 (0 1 1 0 1 -2 -1) g_2 (0 1 -2 3 -1 -2 -1) g_3 (0 0 2 8 3 0 -1) g_4 (0 1 1 3 -2 0 -1) g_5 (0 1 -3 4 4 3 -1) g_6 (0 1 3 4 5 4 1)
Family F13:0b (transpose of F13:0a). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 3$. Maximal levels: $k/3$ from $k = 3$.	
$\begin{pmatrix} 2a+1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2a+1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 2a & 1 & 1 & 1 \\ 0 & 0 & -1 & 2a & -1 & 0 \\ -1 & -1 & -1 & 1 & 2a-1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 2a-1 \end{pmatrix}$	Order (1 0 8 2 -1 -4 0) g_1 (0 1 -2 2 -2 0 1) g_2 (0 1 2 1 5 -4 -1) g_3 (0 1 3 3 0 0 -1) g_4 (0 0 0 6 1 -4 -1) g_5 (0 1 -3 2 4 1 -1) g_6 (0 1 0 0 0 -2 -1)

Table A.27: Degree 13, diameter class $k \equiv 1 \pmod{3}$, $a = (k-1)/3$

LGM	Polynomial in $2a$
Family F13:1a (transpose of F13:1b, conjugate of F13:2a). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 4$. Maximal levels: $(k-1)/3$ from $k = 4$.	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a & -1 & 0 & 0 & 0 \\ 1 & 1 & 2a+1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2a+1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 2a+1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2a+1 \end{pmatrix}$	Order (1 4 16 30 29 16 4) g_1 (0 1 6 21 33 26 9) g_2 (0 1 4 11 13 2 -3) g_3 (0 1 1 4 6 7 5) g_4 (0 1 3 7 6 4 1) g_5 (0 1 3 19 28 20 7) g_6 (0 0 1 0 -8 -8 -1)
Family F13:1b (transpose of F13:1a, conjugate of F13:2b). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 4$. Maximal levels: $(k-1)/3$ from $k = 4$.	
$\begin{pmatrix} 2a & 1 & 1 & 1 & 1 & 0 \\ -1 & 2a & 1 & 0 & 0 & 0 \\ -1 & -1 & 2a+1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 2a+1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2a+1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 2a+1 \end{pmatrix}$	Order (1 4 16 30 29 16 4) g_1 (0 1 5 14 11 -4 -5) g_2 (0 1 6 24 48 40 11) g_3 (0 1 3 4 0 -1 1) g_4 (0 0 1 6 1 -4 -1) g_5 (0 1 2 7 16 16 5) g_6 (0 1 1 11 14 0 -3)

Table A.28: Degree 13, diameter class $k \equiv 2 \pmod{3}$, $a = (k+1)/3$

LGM	Polynomial in $2a$
Family F13:2a (transpose of F13:2b, conjugate of F13:1a). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 2$. Maximal levels: $(k+1)/3$ from $k = 2$.	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a & -1 & 0 & 0 & 0 \\ 1 & 1 & 2a-1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2a-1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 2a-1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order (1 -4 16 -30 29 -16 4) g_1 (0 1 -4 11 -10 1 1) g_2 (0 1 -5 19 -33 25 -7) g_3 (0 0 1 -8 14 -10 3) g_4 (0 0 1 -3 -1 3 -1) g_5 (0 1 -2 10 -18 15 -5) g_6 (0 0 1 1 0 -5 3)
Family F13:2b (transpose of F13:2a, conjugate of F13:1b). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 2$. Maximal levels: $(k+1)/3$ from $k = 2$.	
$\begin{pmatrix} 2a & 1 & 1 & 1 & 1 & 0 \\ -1 & 2a & 1 & 0 & 0 & 0 \\ -1 & -1 & 2a-1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 2a-1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2a-1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 2a-1 \end{pmatrix}$	Order (1 -4 16 -30 29 -16 4) g_1 (0 1 -6 21 -33 26 -9) g_2 (0 1 -4 11 -13 2 3) g_3 (0 1 -1 4 -6 7 -5) g_4 (0 1 -3 7 -6 4 -1) g_5 (0 1 -3 19 -28 20 -7) g_6 (0 0 1 0 -8 8 -1)

A.6 Circulant graph families of degrees 14 and 15

Table A.29: Degree 14, diameter class $k \equiv 0 \pmod{7}$, $a = 2k/7$

LGM odd basis	Polynomial in $2a$
Family F14:0a (transpose of F14:0b, conjugate of F14:6a). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 7$. Maximal levels: $(2k+7)/7$ from $k = 7$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 2a & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2a & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 2a & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order (1 2 15 20 21 12 4 0) /2 g_1 (0 0 1 3 -2 -1 -4 0) /2 g_2 (0 0 1 -2 3 -2 -2 0) /2 g_3 (0 0 1 6 14 15 8 0) /2 g_4 (0 0 0 4 6 -1 0 0) /2 g_5 (0 0 1 -1 -3 -5 -3 -2) /2 g_6 (0 1 3 14 17 16 9 2) /2 g_7 (0 1 1 10 13 10 2 0) /2
Family F14:0b (transpose of F14:0a, conjugate of F14:6b). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 7$. Maximal levels: $(2k+7)/7$ from $k = 7$.	
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 2a+1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2a & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2a & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 2a & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 2a & 1 \\ 0 & -1 & 0 & -1 & -1 & -1 & 2a \end{pmatrix}$	Order (1 2 15 20 21 12 4 0) /2 g_1 (0 3 0 26 27 25 10 0) /2 g_2 (0 0 1 -14 -12 -22 -4 0) /2 g_3 (0 0 6 9 23 21 10 0) /2 g_4 (0 0 4 1 -6 -5 2 0) /2 g_5 (0 1 3 -2 -7 -7 -9 -2) /2 g_6 (0 2 5 13 13 14 3 2) /2 g_7 (0 2 7 43 38 30 4 0) /2

Table A.30: Degree 14, diameter class $k \equiv 1 \pmod{7}$, $a = (2k - 2)/7$

LGM odd basis	Polynomial in $2a$
Family F14:1 (self-transpose, conjugate of F14:5, translate of F15:5). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 8$. Maximal levels: $(2k + 5)/7$ from $k = 7$.	
$\begin{pmatrix} 2a & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a+1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2a+1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a+1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 2a+1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 2a+1 \end{pmatrix}$	Order (1 6 28 76 127 126 67 14) /2 g_1 (0 0 2 13 36 53 41 14) /2 g_2 (0 0 2 10 25 34 22 4) /2 g_3 (0 0 1 7 12 8 -1 -2) /2 g_4 (0 0 2 3 -1 -12 -15 -6) /2 g_5 (0 1 5 15 27 34 -24 8) /2 g_6 (0 1 5 17 33 34 17 2) /2 g_7 (0 0 1 1 -6 -15 -12 -4) /2

Table A.31: Degree 14, diameter class $k \equiv 2 \pmod{7}$, $a = (2k + 3)/7$

LGM odd basis	Polynomial in $2a$
Family F14:2a (transpose of F14:2b, conjugate of F14:4a). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 2$. Maximal levels: $(2k + 3)/7$ from $k = 2$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a-1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 2a-1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 2a-1 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 2a & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order (1 -4 21 -46 50 -30 8 0) /2 g_1 (0 2 -4 30 -55 43 -16 0) /2 g_2 (0 0 1 2 22 -41 16 0) /2 g_3 (0 0 1 4 -18 18 -4 0) /2 g_4 (0 1 -4 14 -10 4 -4 0) /2 g_5 (0 0 1 -11 29 -38 23 -4) /2 g_6 (0 1 -3 10 -17 12 -7 4) /2 g_7 (0 2 -10 49 -101 80 -20 0) /2
Family F14:2b (transpose of F14:2a, conjugate of F14:4b). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 2$. Maximal levels: $(2k + 3)/7$ from $k = 2$.	
$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 1 & -1 & 0 \\ -1 & 2a-1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 2a-1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & 2a-1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 2a & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 2a & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & 2a \end{pmatrix}$	Order (1 -4 21 -46 50 -30 8 0) /2 g_1 (0 1 -5 23 -38 27 -8 0) /2 g_2 (0 0 0 4 -17 21 -8 0) /2 g_3 (0 0 1 -8 14 -12 4 0) /2 g_4 (0 1 -2 10 -14 10 -4 0) /2 g_5 (0 0 1 2 -7 11 -11 4) /2 g_6 (0 1 -5 19 -39 39 -19 4) /2 g_7 (0 1 -2 14 -27 18 -4 0) /2

Table A.32: Degree 14, diameter class $k \equiv 3 \pmod{7}$, $a = (2k + 1)/7$

LGM odd basis	Polynomial in $2a$
Family F14:3 (arc-transitive, self-transpose, self-conjugate, translate of F15:0). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 3$. Maximal levels: $(2k + 8)/7$ from $k = 3$.	
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order (1 0 14 0 21 0 7 0) /2 g_1 (0 1 0 10 0 9 0 1) g_2 (0 0 0 3 -5 3 -4 1) g_3 (0 0 1 2 4 1 1 -1) g_4 (0 0 1 1 2 4 2 1) g_5 (0 0 1 -1 2 -4 2 -1) g_6 (0 0 1 -2 4 -1 1 1) g_7 (0 0 0 3 5 3 4 1)

Note: generator polynomials not divided by 2

Table A.33: Degree 14, diameter class $k \equiv 4 \pmod{7}$, $a = (2k - 1)/7$

LGM odd basis	Polynomial in $2a$
Family F14:4a (transpose of F14:4b, conjugate of F14:2a). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 11$. Maximal levels: $(2k + 6)/7$ from $k = 11$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 2a+1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 2a+1 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 2a & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order (1 4 21 46 50 30 8 0) /2 g_1 (0 1 5 23 38 27 8 0) /2 g_2 (0 0 0 4 17 21 8 0) /2 g_3 (0 0 1 8 14 12 4 0) /2 g_4 (0 1 2 10 14 10 4 0) /2 g_5 (0 0 1 -2 -7 -11 -11 -4) /2 g_6 (0 1 5 19 39 39 19 4) /2 g_7 (0 1 2 14 27 18 4 0) /2
Family F14:4b (transpose of F14:4a, conjugate of F14:2b). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 4$. Maximal levels: $(2k + 6)/7$ from $k = 11$.	
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 2a+1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 2a+1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & 2a+1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 2a & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 2a & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & 2a \end{pmatrix}$	Order (1 4 21 46 50 30 8 0) /2 g_1 (0 2 4 30 55 43 16 0) /2 g_2 (0 0 1 -2 22 41 16 0) /2 g_3 (0 0 1 -4 -18 -18 -4 0) /2 g_4 (0 1 4 14 10 4 4 0) /2 g_5 (0 0 1 11 29 38 23 4) /2 g_6 (0 1 3 10 17 12 7 4) /2 g_7 (0 2 10 49 101 80 20 0) /2

Table A.34: Degree 14, diameter class $k \equiv 5 \pmod{7}$, $a = (2k + 4)/7$

LGM odd basis	Polynomial in $2a$
Family F14:5 (self-transpose, conjugate of F14:1, translate of F15:2). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 5$. Maximal levels: $(2k + 4)/7$ from $k = 5$.	
$\begin{pmatrix} 2a & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a-1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a-1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2a-1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a-1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 2a-1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 2a-1 \end{pmatrix}$	Order (1 -6 28 -76 127 -126 67 -14) /2 g_1 (0 0 2 -13 36 -53 41 -14) /2 g_2 (0 0 2 -10 25 -34 22 -4) /2 g_3 (0 0 1 -7 12 -8 -1 2) /2 g_4 (0 0 2 -3 -1 12 -15 6) /2 g_5 (0 1 -5 15 -27 34 -24 8) /2 g_6 (0 1 -5 17 -33 34 -17 2) /2 g_7 (0 0 1 -1 -6 15 -12 4) /2

Table A.35: Degree 14, diameter class $k \equiv 6 \pmod{7}$, $a = (2k + 2)/7$

LGM odd basis	Polynomial in $2a$
Family F14:6a (transpose of F14:6b, conjugate of F14:0a). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 6$. Maximal levels: $(2k + 2)/7$ from $k = 6$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a-1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 2a & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2a & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 2a & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order (1 -2 15 -20 21 -12 4 0) /2 g_1 (0 3 0 26 -27 25 -10 0) /2 g_2 (0 0 1 14 -12 22 -4 0) /2 g_3 (0 0 6 -9 23 -21 10 0) /2 g_4 (0 0 4 -1 -6 5 2 0) /2 g_5 (0 1 -3 -2 7 -7 9 -2) /2 g_6 (0 2 -5 13 -13 14 -3 2) /2 g_7 (0 2 -7 43 -38 30 -4 0) /2
Family F14:6b (transpose of F14:6a, conjugate of F14:0b). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 13$. Maximal levels: $(2k + 2)/7$ from $k = 6$.	
$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 2a-1 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 2a & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2a & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 2a & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 2a & 1 \\ 0 & -1 & 0 & -1 & -1 & -1 & 2a \end{pmatrix}$	Order (1 -2 15 -20 21 -12 4 0) /2 g_1 (0 0 1 -3 -2 1 -4 0) /2 g_2 (0 0 1 2 3 2 -2 0) /2 g_3 (0 0 1 -6 14 -15 8 0) /2 g_4 (0 0 0 4 -6 -1 0 0) /2 g_5 (0 0 1 1 -3 5 -3 2) /2 g_6 (0 1 -3 14 -17 16 -9 2) /2 g_7 (0 1 -1 10 -13 10 -2 0) /2

Table A.36: Degree 15, diameter class $k \equiv 0 \pmod{7}$, $a = 2k/7$

LGM	Polynomial in $2a$
Family F15:0 (self-transpose, self-conjugate, translate of F14:3). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 7$. Maximal levels: $(2k + 7)/7$ from $k = 7$.	
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order (1 0 14 0 21 0 7 0) g_1 (0 1 4 10 2 9 -2 1) g_2 (0 1 4 4 12 3 6 -1) g_3 (0 1 2 0 4 1 4 1) g_4 (0 1 0 -2 0 -7 0 -1) g_5 (0 1 -2 0 -4 1 -4 1) g_6 (0 1 -4 4 -12 3 -6 -1) g_7 (0 1 -4 10 -2 9 2 1)

Table A.37: Degree 15, diameter class $k \equiv 1 \pmod{7}$, $a = (2k - 2)/7$

LGM	Polynomial in $2a$
Family F15:1 (self-transpose, conjugate of F15:6). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 8$. Maximal levels: $(2k + 5)/7$ from $k = 8$.	
$\begin{pmatrix} 2a & -1 & -1 & 0 & 0 & -1 & -1 \\ 1 & 2a & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 2a & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2a+1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2a+1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 2a+1 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 & 2a+1 \end{pmatrix}$	Order (1 4 20 44 57 44 19 4) g_1 (0 1 6 31 60 61 32 5) g_2 (0 1 6 20 34 34 21 5) g_3 (0 1 3 6 1 -8 -12 -7) g_4 (0 1 0 6 13 17 10 3) g_5 (0 1 3 9 23 29 20 5) g_6 (0 0 0 8 15 16 6 -1) g_7 (0 0 2 -2 -16 -22 -17 -3)

Table A.38: Degree 15, diameter class $k \equiv 2 \pmod{7}$, $a = (2k + 3)/7$

LGM	Polynomial in $2a$
Family F15:2 (self-transpose, conjugate of F15:5, translate of F14:5). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 2$. Maximal levels: $(2k + 3)/7$ from $k = 2$.	
$\begin{pmatrix} 2a & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a-1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a-1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2a-1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a-1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 2a-1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 2a-1 \end{pmatrix}$	Order (1 -6 28 -76 127 -126 67 -14) g_1 (0 1 -6 18 -40 59 -52 21) g_2 (0 1 -4 10 -8 -11 26 -11) g_3 (0 1 -4 12 -20 27 -16 1) g_4 (0 1 -8 28 -58 65 -38 5) g_5 (0 1 -8 38 -84 107 -70 19) g_6 (0 1 -2 8 -12 9 -4 -1) g_7 (0 1 -4 22 -62 89 -58 11)

Table A.39: Degree 15, diameter class $k \equiv 3 \pmod{7}$, $a = (2k + 1)/7$

LGM	Polynomial in $2a$
Family F15:3a (transpose of F15:3b). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 3$. Maximal levels: $(2k + 1)/7$ from $k = 3$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 2a-1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a-1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 2a & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2a & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 2a+1 \end{pmatrix}$	Order (1 -2 14 -16 11 -6 3 -2) g_1 (0 1 2 16 0 -5 -4 -1) g_2 (0 1 2 4 -14 1 -2 3) g_3 (0 1 0 -4 2 3 0 1) g_4 (0 1 -2 0 -4 3 2 -1) g_5 (0 1 -4 8 8 -3 -6 -1) g_6 (0 1 6 24 -30 7 -4 5) g_7 (0 1 -4 10 -2 -5 2 -1)
Family F15:3b (transpose of F15:3a). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 3$. Maximal levels: $(2k + 1)/7$ from $k = 3$.	
$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2a-1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 2a-1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 2a & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 2a & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 2a & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 2a+1 \end{pmatrix}$	Order (1 -2 14 -16 11 -6 3 -2) g_1 (0 1 -4 13 -11 -1 1 1) g_2 (0 1 -3 5 -4 5 -4 1) g_3 (0 1 0 2 -1 3 -5 1) g_4 (0 1 0 1 6 -16 12 -3) g_5 (0 0 3 -6 6 -5 1 1) g_6 (0 0 3 2 6 -9 2 -1) g_7 (0 1 -1 9 -20 20 -9 1)

Table A.40: Degree 15, diameter class $k \equiv 4 \pmod{7}$, $a = (2k - 1)/7$

LGM	Polynomial in $2a$
Family F15:4 (self-transpose). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 4$. Maximal levels: $(2k + 6)/7$ from $k = 4$.	
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 2a & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 2a & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2a & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2a & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order (1 2 14 20 27 18 11 2) g_1 (0 1 4 12 14 11 6 3) g_2 (0 1 4 16 16 19 8 5) g_3 (0 1 2 4 2 7 6 5) g_4 (0 1 0 0 -8 -11 -10 -3) g_5 (0 1 0 2 4 3 2 -1) g_6 (0 1 4 8 14 13 6 1) g_7 (0 1 -2 8 6 15 6 7)

Table A.41: Degree 15, diameter class $k \equiv 5 \pmod{7}$, $a = (2k - 3)/7$

LGM	Polynomial in $2a$
Family F15:5 (self-transpose, conjugate of F15:2, translate of F14:1). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 12$. Maximal levels: $(2k + 4)/7$ from $k = 5$.	
$\begin{pmatrix} 2a & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a+1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 2a+1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a+1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 2a+1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 2a+1 \end{pmatrix}$	Order (1 6 28 76 127 126 67 14) g_1 (0 1 6 18 40 59 52 21) g_2 (0 1 4 10 8 -11 -26 -11) g_3 (0 1 4 12 20 27 16 1) g_4 (0 1 8 28 58 65 38 5) g_5 (0 1 8 38 84 107 70 19) g_6 (0 1 2 8 12 9 4 -1) g_7 (0 1 4 22 62 89 58 11)

Table A.42: Degree 15, diameter class $k \equiv 6 \pmod{7}$, $a = (2k + 2)/7$

LGM	Polynomial in $2a$
Family F15:6 (self-transpose, conjugate of F15:1). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 6$. Maximal levels: $(2k + 2)/7$ from $k = 6$.	
$\begin{pmatrix} 2a & -1 & -1 & 0 & 0 & -1 & -1 \\ 1 & 2a & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 2a & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2a-1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2a-1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 2a-1 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 & 2a-1 \end{pmatrix}$	Order (1 -4 20 -44 57 -44 19 -4) g_1 (0 1 -6 31 -60 61 -32 5) g_2 (0 1 -6 20 -34 34 -21 5) g_3 (0 1 -3 6 -1 -8 12 -7) g_4 (0 1 0 6 -13 17 -10 3) g_5 (0 1 -3 9 -23 29 -20 5) g_6 (0 0 0 8 -15 16 -6 -1) g_7 (0 0 2 2 -16 22 -17 3)

A.7 Circulant graph families of degrees 16 and 17

Table A.43: Degree 16, diameter class $k \equiv 0 \pmod{4}$, $a = k/4$

LGM odd basis	Polynomial in $2a$									
Family F16:0a (transpose of F16:0b, conjugate of F16:3a). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 12$. Maximal levels: $(k + 4)/4$ from $k = 8$.										
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 2a & 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2a & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 2a & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2a & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 & -1 & 2a-1 \end{pmatrix}$	Order	(1	2	20	28	11	2	-4	-4	0)/2
	g_1	(0	1	3	22	14	-3	-7	-6	0)/2
	g_2	(0	0	2	3	5	-5	-7	-2	0)/2
	g_3	(0	0	1	-6	-4	-8	-11	0	0)/2
	g_4	(0	0	1	10	18	12	7	2	-2)/2
	g_5	(0	1	1	10	10	-1	-5	-6	-2)/2
	g_6	(0	0	0	5	25	27	5	-2	0)/2
	g_7	(0	1	0	15	15	-14	-13	0	0)/2
	g_8	(0	1	4	19	37	36	19	4	0)/2
Family F16:0b (transpose of F16:0a, conjugate of F16:3b). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 12$. Maximal levels: $(k + 4)/4$ from $k = 8$.										
$\begin{pmatrix} 2a+1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 2a+1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 2a+1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 2a & 0 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 2a & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 2a & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 2a & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$	Order	(1	2	20	28	11	2	-4	-4	0)/2
	g_1	(0	2	0	34	21	-8	-3	-2	0)/2
	g_2	(0	0	2	4	-3	12	3	-6	0)/2
	g_3	(0	0	1	11	12	11	1	-8	0)/2
	g_4	(0	0	1	-6	-22	-14	3	4	2)/2
	g_5	(0	1	3	14	6	-3	5	0	-2)/2
	g_6	(0	0	1	-6	22	40	-3	-14	0)/2
	g_7	(0	2	6	45	63	9	-21	-8	0)/2
	g_8	(0	1	2	10	15	3	-7	-4	0)/2

Table A.44: Degree 16, diameter class $k \equiv 1 \pmod{4}$, $a = (k - 1)/4$

Family F16:1a (transpose of F16:1b, conjugate of F16:2a). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 9$. Maximal levels: $(k + 3)/4$ from $k = 5$.

LGM odd basis

$$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a+1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 2a+1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2a+1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 2a+1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2a & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$$

Polynomial in $2a$

Order	(1	6	33	100	183	212	151	60	10)/2
g_1	(0	1	7	39	100	151	137	65	12)/2
g_2	(0	0	2	3	6	4	-8	-11	-4)/2
g_3	(0	0	1	0	-16	-49	-63	-37	-8)/2
g_4	(0	0	0	4	14	9	-6	-7	-2)/2
g_5	(0	0	0	8	26	46	43	21	4)/2
g_6	(0	2	10	48	118	176	159	79	16)/2
g_7	(0	0	2	13	43	82	85	45	10)/2
g_8	(0	1	4	21	61	99	87	37	6)/2

Family F16:1b (transpose of F16:1a, conjugate of F16:2b). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 9$. Maximal levels: $(k + 3)/4$ from $k = 5$.

LGM odd basis

$$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 2a+1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 2a+1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 2a+1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 2a+1 & 1 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 2a & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 & 0 & 2a \end{pmatrix}$$

Polynomial in $2a$

Order	(1	6	33	100	183	212	151	60	10)/2
g_1	(0	2	9	56	157	251	236	115	22)/2
g_2	(0	0	2	12	37	65	65	33	6)/2
g_3	(0	0	0	1	-23	-71	-100	-71	-18)/2
g_4	(0	0	0	4	8	0	-21	-25	-8)/2
g_5	(0	0	1	12	46	90	103	65	16)/2
g_6	(0	1	4	16	26	21	-2	-15	-6)/2
g_7	(0	1	7	34	91	149	149	81	18)/2
g_8	(0	2	13	67	188	301	290	153	32)/2

Table A.45: Degree 16, diameter class $k \equiv 2 \pmod{4}$, $a = (k+2)/4$

Family F16:2a (transpose of F16:2b, conjugate of F16:1a). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 10$. Maximal levels: $(k+2)/4$ from $k = 6$.

LGM odd basis

$$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 2a-1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a-1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 2a-1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 2a-1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2a & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$$

Polynomial in $2a$

Order	(1	-6	33	-100	183	-212	151	-60	10)/2
g_1	(0	2	-9	56	-157	251	-236	115	-22)/2
g_2	(0	0	2	-12	37	-65	65	-33	6)/2
g_3	(0	0	0	1	23	-71	100	-71	18)/2
g_4	(0	0	0	4	-8	0	21	-25	8)/2
g_5	(0	0	1	-12	46	-90	103	-65	16)/2
g_6	(0	1	-4	16	-26	21	2	-15	6)/2
g_7	(0	1	-7	34	-91	149	-149	81	-18)/2
g_8	(0	2	-13	67	-188	301	-290	153	-32)/2

Family F16:2b (transpose of F16:2a, conjugate of F16:1b). Graphs are largest known from $k = 6$. odd-girth maximum from $k = 10$. Maximal levels: $(k+2)/4$ from $k = 6$.

LGM odd basis

$$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2a-1 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 2a-1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 2a-1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 2a-1 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 2a-1 & 1 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 2a & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 & 0 & 2a \end{pmatrix}$$

Polynomial in $2a$

Order	(1	-6	33	-100	183	-212	151	-60	10)/2
g_1	(0	1	-7	39	-100	151	-137	65	-12)/2
g_2	(0	0	2	-3	6	-4	-8	11	-4)/2
g_3	(0	0	1	0	-16	49	-63	37	-8)/2
g_4	(0	0	0	4	-14	9	6	-7	2)/2
g_5	(0	0	0	8	-26	46	-43	21	-4)/2
g_6	(0	2	-10	48	-118	176	-159	79	-16)/2
g_7	(0	0	2	-13	43	-82	85	-45	10)/2
g_8	(0	1	-4	21	-61	99	-87	37	-6)/2

Table A.46: Degree 16, diameter class $k \equiv 3 \pmod{4}$, $a = (k+1)/4$

LGM odd basis	Polynomial in $2a$
Family F16:3a (transpose of F16:3b, conjugate of F16:0a). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 11$. Maximal levels: $(k+5)/4$ from $k = 11$.	
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 2a-1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a-1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 2a & 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2a & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 2a & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2a & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 & -1 & 2a+1 \end{pmatrix}$	Order (1 -2 20 -28 11 -2 -4 4 0)/2 g_1 (0 2 0 34 -21 -8 3 -2 0)/2 g_2 (0 0 2 -4 -3 -12 3 6 0)/2 g_3 (0 0 1 -11 12 -11 1 8 0)/2 g_4 (0 0 1 6 -22 14 3 -4 2)/2 g_5 (0 1 -3 14 -6 -3 -5 0 2)/2 g_6 (0 0 1 6 22 -40 -3 14 0)/2 g_7 (0 2 -6 45 -63 9 21 -8 0)/2 g_8 (0 1 -2 10 -15 3 7 -4 0)/2
Family F16:3b (transpose of F16:3a, conjugate of F16:0b). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 11$. Maximal levels: $(k+5)/4$ from $k = 11$.	
$\begin{pmatrix} 2a-1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 2a-1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 2a-1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 2a & 0 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 2a & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & 2a & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 2a & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 & 1 & 2a+1 \end{pmatrix}$	Order (1 -2 20 -28 11 -2 -4 4 0)/2 g_1 (0 1 -3 22 -14 -3 7 -6 0)/2 g_2 (0 0 2 -3 5 5 -7 2 0)/2 g_3 (0 0 1 6 -4 8 -11 0 0)/2 g_4 (0 0 1 -10 18 -12 7 -2 -2)/2 g_5 (0 1 -1 10 -10 -1 5 -6 2)/2 g_6 (0 0 0 5 -25 27 -5 -2 0)/2 g_7 (0 1 0 15 -15 -14 13 0 0)/2 g_8 (0 1 -4 19 -37 36 -19 4 0)/2

Table A.47: Degree 17, diameter class $k \equiv 0 \pmod{4}$, $a = k/4$

LGM	Polynomial in $2a$
Family F17:0a (transpose of F17:0b). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 12$. Maximal levels: $(k+4)/4$ from $k = 8$.	
$\begin{pmatrix} 2a+1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 2a+1 & -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 1 & a2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & a2 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 2a & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 2a & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 2a-1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & -1 & 2a-1 \end{pmatrix}$	Order (1 0 17 0 4 6 -4 0 0) g_1 (0 1 -1 16 -15 1 -1 -10 1) g_2 (0 1 -1 17 -13 1 5 -11 1) g_3 (0 1 -1 18 -18 8 10 -11 1) g_4 (0 1 0 18 -1 7 0 -10 1) g_5 (0 1 0 16 -3 2 0 -9 1) g_6 (0 1 1 17 14 2 14 8 -1) g_7 (0 0 1 0 15 12 13 8 -1) g_8 (0 1 1 17 21 9 15 9 -1)
Family F17:0b (transpose of F17:0a). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 12$. Maximal levels: $(k+4)/4$ from $k = 8$.	
$\begin{pmatrix} 2a+1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 2a+1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & a2 & 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & a2 & -1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 2a & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 2a & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & -1 & 2a-1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 0 & 1 & 2a-1 \end{pmatrix}$	Order (1 0 17 0 4 6 -4 0 0) g_1 (0 1 -4 15 -13 10 4 -6 1) g_2 (0 1 -1 3 -2 -1 4 -5 1) g_3 (0 1 2 22 -6 -12 7 3 -1) g_4 (0 0 4 4 18 24 -5 -6 1) g_5 (0 1 3 8 -1 4 -5 -3 1) g_6 (0 1 1 4 23 -9 -9 6 -1) g_7 (0 1 -2 20 17 3 -4 -4 1) g_8 (0 1 1 3 4 1 4 3 -1)

Table A.48: Degree 17, diameter class $k \equiv 1 \pmod{4}$, $a = (k - 1)/4$

LGM	Polynomial in $2a$									
Family F17:1a (transpose of F17:1b, conjugate of F17:3a). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 13$. Maximal levels: $(k + 3)/4$ from $k = 5$.										
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 2a+1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2a & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2a & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 & 2a & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2a \end{pmatrix}$	Order	(1	4	25	60	82	78	50	20	4)
	g_1	(0	1	7	30	53	62	49	25	7)
	g_2	(0	0	4	8	15	18	15	9	3)
	g_3	(0	1	4	10	14	11	7	2	-1)
	g_4	(0	1	1	4	6	6	5	2	1)
	g_5	(0	1	2	8	20	27	24	13	3)
	g_6	(0	1	3	9	20	26	22	14	5)
	g_7	(0	0	2	-8	-27	-37	-34	-19	-5)
	g_8	(0	0	3	10	15	13	6	0	-1)
Family F17:1b (transpose of F17:1a, conjugate of F17:3b). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 13$. Maximal levels: $(k + 3)/4$ from $k = 5$.										
$\begin{pmatrix} 2a+1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2a+1 & 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 2a+1 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & 0 & 2a & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2a & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & 2a & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & -1 & 2a \end{pmatrix}$	Order	(1	4	25	60	82	78	50	20	4)
	g_1	(0	1	-2	0	0	0	2	2	1)
	g_2	(0	0	5	22	38	44	32	14	3)
	g_3	(0	1	3	1	1	7	11	9	3)
	g_4	(0	1	7	17	24	26	19	9	3)
	g_5	(0	1	7	24	36	26	9	0	-1)
	g_6	(0	1	7	21	40	49	38	17	3)
	g_7	(0	1	3	-3	-19	-32	-31	-18	-5)
	g_8	(0	1	0	5	24	36	28	13	3)

Table A.49: Degree 17, diameter class $k \equiv 2 \pmod{4}$, $a = (k - 2)/4$

Family F17:2a (transpose of F17:2b). Graphs are largest known from $k = 6$.
odd-girth maximum from $k = 14$. Maximal levels: $(k + 2)/4$ from $k = 6$.

LGM

$$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2a+1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 2a+1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 2a+1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 & 2a+2 \end{pmatrix}$$

Polynomial in $2a$

Order	(1	8	44	154	340	476	412	200	40)
g_1	(0	0	2	24	95	190	212	124	29)
g_2	(0	1	3	9	34	82	111	79	23)
g_3	(0	1	4	12	26	37	30	10	1)
g_4	(0	1	6	14	9	-23	-56	-50	-17)
g_5	(0	1	6	19	42	64	62	33	7)
g_6	(0	1	9	50	146	242	231	117	25)
g_7	(0	1	11	50	134	222	224	123	25)
g_8	(0	1	11	52	134	210	204	116	31)

Family F17:2b (transpose of F17:2a). Graphs are largest known from $k = 6$.
odd-girth maximum from $k = 14$. Maximal levels: $(k + 2)/4$ from $k = 6$.

LGM

$$\begin{pmatrix} 2a & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2a+1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 2a+1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 2a+1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2a+1 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 2a+1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 2a+1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 0 & 1 & 2a+2 \end{pmatrix}$$

Polynomial in $2a$

Order	(1	8	44	154	340	476	412	200	40)
g_1	(0	1	5	13	24	38	49	42	17)
g_2	(0	1	4	11	24	35	29	9	-1)
g_3	(0	1	7	24	41	31	-5	-26	-13)
g_4	(0	1	10	50	152	277	303	182	43)
g_5	(0	1	10	52	143	231	222	117	27)
g_6	(0	0	1	-2	-26	-69	-89	-55	-11)
g_7	(0	0	2	11	21	15	-4	-9	-1)
g_8	(0	0	1	-5	-29	-58	-57	-26	-5)

Table A.50: Degree 17, diameter class $k \equiv 3 \pmod{4}$, $a = (k + 1)/4$

LGM	Polynomial in $2a$									
Family F17:3a (transpose of F17:3b, conjugate of F17:1a). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 11$. Maximal levels: $(k + 5)/4$ from $k = 11$.										
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 2a-1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 2a-1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2a & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2a & 1 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 & 2a & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2a \end{pmatrix}$	Order	(1	-4	25	-60	82	-78	50	-20	4)
	g_1	(0	1	2	0	0	0	-2	2	-1)
	g_2	(0	0	5	-22	38	-44	32	-14	3)
	g_3	(0	1	-3	1	-1	7	-11	9	-3)
	g_4	(0	1	-7	17	-24	26	-19	9	-3)
	g_5	(0	1	-7	24	-36	26	-9	0	1)
	g_6	(0	1	-7	21	-40	49	-38	17	-3)
	g_7	(0	1	-3	-3	19	-32	31	-18	5)
	g_8	(0	1	0	5	-24	36	-28	13	-3)
Family F17:3b (transpose of F17:3a, conjugate of F17:1b). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 11$. Maximal levels: $(k + 5)/4$ from $k = 11$.										
$\begin{pmatrix} 2a-1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2a-1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2a-1 & 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 2a-1 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & 0 & 2a & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2a & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 & 2a & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & -1 & 2a \end{pmatrix}$	Order	(1	-4	25	-60	82	-78	50	-20	4)
	g_1	(0	1	-7	30	-53	62	-49	25	-7)
	g_2	(0	0	4	-8	15	-18	15	-9	3)
	g_3	(0	1	-4	10	-14	11	-7	2	1)
	g_4	(0	1	-1	4	-6	6	-5	2	-1)
	g_5	(0	1	-2	8	-20	27	-24	13	-3)
	g_6	(0	1	-3	9	-20	26	-22	14	-5)
	g_7	(0	0	2	8	-27	37	-34	19	-5)
	g_8	(0	0	3	-10	15	-13	6	0	-1)

A.8 Circulant graph families of degrees 18 and 19

Table A.51: Degree 18, diameter class $k \equiv 0 \pmod{9}$, $a = 2k/9$

LGM odd basis	Polynomial in $2a$
Family F18:0a (transpose of F18:0b, conjugate of F18:8a). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 9$. Maximal levels: $(2k + 9)/9$ from $k = 9$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 2a & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2a & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order (1 2 23 34 59 52 35 16 4 0)/2 g_1 (0 0 1 7 16 11 12 5 0 0)/2 g_2 (0 0 1 2 -6 -5 -8 -4 -2 0)/2 g_3 (0 0 2 4 12 9 10 9 4 0)/2 g_4 (0 0 1 4 3 8 11 3 0 0)/2 g_5 (0 0 1 -5 -8 -24 -24 -19 -9 -2)/2 g_6 (0 1 3 18 26 35 28 16 7 2)/2 g_7 (0 0 1 -3 6 11 11 8 2 0)/2 g_8 (0 0 1 7 19 24 15 6 2 0)/2 g_9 (0 1 1 16 20 23 21 10 2 0)/2
Family F18:0b (transpose of F18:0a, conjugate of F18:8b). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 18$. Maximal levels: $(2k + 9)/9$ from $k = 9$.	
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 2a & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2a & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 2a & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 2a & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & 2a & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 2a \end{pmatrix}$	Order (1 2 23 34 59 52 35 16 4 0)/2 g_1 (0 1 2 15 15 24 15 7 2 0)/2 g_2 (0 0 2 1 6 4 2 4 0 0)/2 g_3 (0 0 2 4 5 2 -2 -5 -2 0)/2 g_4 (0 0 2 2 11 12 15 7 2 0)/2 g_5 (0 0 1 10 16 27 25 18 9 2)/2 g_6 (0 1 1 13 18 32 27 17 7 2)/2 g_7 (0 0 0 7 5 14 10 4 0 0)/2 g_8 (0 0 1 -4 -2 -9 -1 2 0 0)/2 g_9 (0 0 1 -2 -8 -8 -11 -6 0 0)/2

Table A.52: Degree 18, diameter class $k \equiv 1 \pmod{9}$, except $k \equiv 1 \pmod{4}5$,
 $a = (2k - 2)/9$

LGM odd basis	Polynomial in $2a$
Family F18:1a (transpose of F18:1b). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 19$. Maximal levels: $(2k + 7)/9$ from $k = 10$.	
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2a & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 2a & -1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2a+1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 2a+1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 2a+1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & -1 & 0 & 2a+1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2a+1 \end{pmatrix}$	Order (1 6 37 126 265 346 267 112 20 0)/2 g_1 (0 1 9 42 117 210 243 169 63 10)/2 g_2 (0 0 3 5 -9 -55 -103 -98 -49 -10)/2 g_3 (0 0 0 17 71 149 167 90 18 0)/2 g_4 (0 0 3 20 56 89 78 37 8 0)/2 g_5 (0 1 2 12 38 82 97 55 12 0)/2 g_6 (0 0 3 18 46 71 59 23 4 0)/2 g_7 (0 0 1 14 51 103 123 76 18 0)/2 g_8 (0 2 10 48 131 227 232 122 26 0)/2 g_9 (0 0 1 -2 -23 -66 -90 -58 -14 0)/2
Family F18:1b (transpose of F18:1a). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 19$. Maximal levels: $(2k + 7)/9$ from $k = 10$.	
$\begin{pmatrix} 2a & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2a & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2a & 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 2a+1 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & 0 & 2a+1 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 2a+1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 2a+1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 1 & 0 & 2a+1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2a+1 \end{pmatrix}$	Order (1 6 37 126 265 346 267 112 20 0)/2 g_1 (0 1 5 34 117 234 285 201 73 10)/2 g_2 (0 0 1 3 9 31 61 66 39 10)/2 g_3 (0 0 2 11 51 127 179 134 40 0)/2 g_4 (0 0 1 8 36 87 92 41 6 0)/2 g_5 (0 1 4 24 58 84 83 51 14 0)/2 g_6 (0 0 1 2 20 77 113 73 18 0)/2 g_7 (0 0 1 2 27 61 65 46 16 0)/2 g_8 (0 0 0 0 13 53 76 44 8 0)/2 g_9 (0 2 11 64 193 308 254 92 8 0)/2

Table A.53: Degree 18, diameter class $k \equiv 1 \pmod{9}$, $a = (2k - 2)/9$

LGM odd basis	Polynomial in $2a$
Family F18:1c (transpose of F18:1d, conjugate of F18:7a). Largest known from $k = 46$ for $k \equiv 1 \pmod{4}5$. odd-girth maximum from $k = 19$. Maximal levels: $(2k + 7)/9$ from $k = 10$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 2a & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 2a+1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 2a+1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 2a+1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 2a+1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2a+1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2a+1 \end{pmatrix}$	Order (1 6 37 122 251 342 305 172 56 8)/2 g_1 (0 1 6 41 144 281 334 238 91 14)/2 g_2 (0 0 0 3 -6 -42 -71 -55 -19 -2)/2 g_3 (0 0 1 10 27 43 39 10 -8 -4)/2 g_4 (0 0 1 2 -7 -42 -77 -65 -26 -4)/2 g_5 (0 0 2 7 16 33 42 29 11 2)/2 g_6 (0 0 1 -2 -2 8 28 33 15 2)/2 g_7 (0 1 6 30 88 169 209 157 63 10)/2 g_8 (0 1 6 31 92 152 142 76 24 4)/2 g_9 (0 1 3 18 36 28 -6 -33 -25 -6)/2
Family F18:1d (transpose of F18:1c, conjugate of F18:7b). Largest known from $k = 46$ for $k \equiv 1 \pmod{4}5$. odd-girth maximum from $k = 19$. Maximal levels: $(2k + 7)/9$ from $k = 10$.	
$\begin{pmatrix} 2a & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2a & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 2a & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 2a+1 & 0 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 & 2a+1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 2a+1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 2a+1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 2a+1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 2a+1 \end{pmatrix}$	Order (1 6 37 122 251 342 305 172 56 8)/2 g_1 (0 2 12 76 221 358 345 183 43 2)/2 g_2 (0 0 1 13 69 177 231 156 51 6)/2 g_3 (0 0 0 11 49 112 166 154 78 16)/2 g_4 (0 0 1 0 -16 -12 34 53 26 4)/2 g_5 (0 0 2 1 -10 -38 -64 -44 -7 2)/2 g_6 (0 0 2 12 28 36 28 18 9 2)/2 g_7 (0 1 3 20 62 125 153 106 39 6)/2 g_8 (0 1 3 20 49 53 36 26 16 4)/2 g_9 (0 2 12 63 208 403 473 328 121 18)/2

Table A.54: Degree 18, diameter class $k \equiv 2 \pmod{9}$, $a = (2k + 5)/9$

Family F18:2a (transpose of F18:2b, conjugate of F18:6a, translate of F19:7a).

Graphs are largest known from $k = 11$. odd-girth maximum from $k = 2$.Maximal levels: $(2k + 5)/9$ from $k = 11$.

LGM odd basis										
	$2a$	-1	-1	-1	-1	0	0	0	0	
	1	$2a$	0	0	-1	-1	-1	-1	-1	
	1	0	$2a - 1$	0	-1	-1	-1	-1	0	
	1	0	0	$2a - 1$	-1	-1	-1	0	-1	
	1	1	1	1	$2a - 1$	-1	-1	0	-1	
	0	1	1	1	1	$2a - 1$	0	1	0	
	0	1	1	1	1	0	$2a - 1$	1	0	
	0	1	1	0	0	-1	-1	$2a - 1$	-1	
	0	1	0	1	1	0	0	1	$2a - 2$	
Polynomial in $2a$										
Order	(1	-8	50	-194	462	-698	672	-394	125	-16)/2
g_1	(0	0	0	7	-36	79	-96	68	-26	4)/2
g_2	(0	0	2	-9	18	-15	-4	19	-15	4)/2
g_3	(0	0	1	-2	-14	57	-92	75	-29	4)/2
g_4	(0	0	2	-9	23	-44	59	-49	22	-4)/2
g_5	(0	0	2	-16	52	-94	105	-71	26	-4)/2
g_6	(0	0	0	9	-41	90	-113	82	-30	4)/2
g_7	(0	1	-7	34	-110	221	-274	203	-79	12)/2
g_8	(0	0	2	-14	52	-107	127	-86	30	-4)/2
g_9	(0	1	-6	30	-86	145	-148	89	-29	4)/2

Family F18:2b (transpose of F18:2a, conjugate of F18:6b, translate of F19:7b).

Graphs are largest known from $k = 11$. odd-girth maximum from $k = 20$.Maximal levels: $(2k + 5)/9$ from $k = 11$.

LGM odd basis										
	$2a$	1	1	1	1	0	0	0	0	
	-1	$2a$	0	0	1	1	1	1	1	
	-1	0	$2a - 1$	0	1	1	1	1	0	
	-1	0	0	$2a - 1$	1	1	1	0	1	
	-1	-1	-1	-1	$2a - 1$	1	1	0	1	
	0	-1	-1	-1	-1	$2a - 1$	0	-1	0	
	0	-1	-1	-1	-1	0	$2a - 1$	-1	0	
	0	-1	-1	0	0	1	1	$2a - 1$	1	
	0	-1	0	-1	-1	0	0	-1	$2a - 2$	
Polynomial in $2a$										
Order	(1	-8	50	-194	462	-698	672	-394	125	-16)/2
g_1	(0	1	-8	55	-211	453	-569	419	-168	28)/2
g_2	(0	0	2	-12	34	-61	83	-84	50	-12)/2
g_3	(0	0	1	-6	1	21	-34	29	-16	4)/2
g_4	(0	0	1	-2	-16	95	-205	213	-106	20)/2
g_5	(0	0	1	3	-28	74	-97	68	-25	4)/2
g_6	(0	0	1	-13	52	-111	143	-106	37	-4)/2
g_7	(0	1	-6	30	-99	200	-244	179	-72	12)/2
g_8	(0	1	-9	48	-172	383	-502	369	-138	20)/2
g_9	(0	1	-5	27	-81	138	-147	95	-32	4)/2

Table A.55: Degree 18, diameter class $k \equiv 3 \pmod{9}$, except $k \equiv 21 \pmod{27}$,
 $a = (2k + 3)/9$

Family F18:3a (transpose of F18:3b, conjugate of F18:5a). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 30$. Maximal levels: $(2k + 12)/9$ from $k = 12$.										
LGM odd basis										
	$2a - 1$	-1	-1	0	0	0	-1	-1	-1	
	1	$2a - 1$	0	-1	-1	-1	-1	-1	-1	
	1	0	$2a - 1$	-1	-1	0	-1	-1	0	
	0	1	1	$2a - 1$	0	0	1	1	1	
	0	1	1	0	$2a$	0	1	1	0	
	0	1	0	0	0	$2a$	1	1	1	
	1	1	1	-1	-1	-1	$2a$	0	0	
	1	1	1	-1	-1	-1	0	$2a$	0	
	1	1	0	-1	0	-1	0	0	$2a$	
Polynomial in $2a$										
Order	$(1$	-4	29	-74	115	-122	81	-34	8	$0)/2$
g_1	$(0$	3	-4	56	-109	142	-119	60	-20	$0)/2$
g_2	$(0$	0	2	-36	88	-131	112	-51	16	$0)/2$
g_3	$(0$	0	0	22	-25	-4	9	-6	4	$0)/2$
g_4	$(0$	0	5	-22	35	-59	46	-18	4	$0)/2$
g_5	$(0$	0	3	-16	7	25	-32	13	0	$0)/2$
g_6	$(0$	0	5	-27	79	-119	101	-55	16	$0)/2$
g_7	$(0$	1	-7	18	-7	-10	16	-18	11	$-4)/2$
g_8	$(0$	2	-11	47	-81	105	-106	63	-23	$4)/2$
g_9	$(0$	2	-11	50	-119	137	-77	22	-4	$0)/2$
Family F18:3b (transpose of F18:3a, conjugate of F18:5b). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 30$. Maximal levels: $(2k + 12)/9$ from $k = 12$.										
LGM odd basis										
	$2a - 1$	1	1	0	0	0	1	1	1	
	-1	$2a - 1$	0	1	1	1	1	1	1	
	-1	0	$2a - 1$	1	1	0	1	1	0	
	0	-1	-1	$2a - 1$	0	0	-1	-1	-1	
	0	-1	-1	0	$2a$	0	-1	-1	0	
	0	-1	0	0	0	$2a$	-1	-1	-1	
	-1	-1	-1	1	1	1	$2a$	0	0	
	-1	-1	-1	1	1	1	0	$2a$	0	
	-1	-1	0	1	0	1	0	0	$2a$	
Polynomial in $2a$										
Order	$(1$	-4	29	-74	115	-122	81	-34	8	$0)/2$
g_1	$(0$	1	-6	34	-69	88	-67	32	-10	$0)/2$
g_2	$(0$	0	2	6	-8	15	-20	7	-2	$0)/2$
g_3	$(0$	0	0	8	-39	48	-17	-2	2	$0)/2$
g_4	$(0$	0	3	-8	21	-21	12	-6	2	$0)/2$
g_5	$(0$	0	1	-4	-15	59	-78	51	-14	$0)/2$
g_6	$(0$	0	3	-11	37	-81	81	-35	6	$0)/2$
g_7	$(0$	0	1	-15	43	-65	68	-45	17	$-4)/2$
g_8	$(0$	1	-3	14	-31	50	-54	36	-17	$4)/2$
g_9	$(0$	2	-7	44	-101	111	-69	22	-2	$0)/2$

Table A.56: Degree 18, diameter class $k \equiv 3 \pmod{9}$, $a = (2k + 3)/9$

Family F18:3c (transpose of F18:3d). Largest known from $k = 21$ for $k \equiv 21 \pmod{2}$.
 odd-girth maximum from $k = 21$. Maximal levels: $(2k + 12)/9$ from $k = 12$.

LGM odd basis									
	$2a - 1$	-1	-1	0	0	0	-1	-1	-1
	1	$2a - 1$	-1	0	-1	-1	-1	-1	-1
	1	1	$2a - 1$	-1	-1	-1	0	0	0
	0	0	1	$2a - 1$	1	1	0	1	1
	0	1	1	-1	$2a$	0	1	0	0
	0	1	1	-1	0	$2a$	1	1	1
	1	1	0	0	-1	-1	$2a$	0	0
	1	1	0	-1	0	-1	0	$2a$	0
	1	1	0	-1	0	-1	0	0	$2a$
Polynomial in $2a$									
Order	(1	-4	28	-70	118	-132	96	-42	8 0)/2
g_1	(0	0	2	-4	12	-10	3	2	-2 0)/2
g_2	(0	0	2	-11	26	-40	37	-22	6 0)/2
g_3	(0	0	0	8	-19	31	-32	20	-6 0)/2
g_4	(0	0	1	-6	6	1	-6	6	-2 0)/2
g_5	(0	0	1	-1	-4	2	4	-5	2 0)/2
g_6	(0	0	2	-5	10	-16	14	-7	2 0)/2
g_7	(0	1	-4	21	-49	74	-68	38	-10 0)/2
g_8	(0	0	0	7	-26	54	-65	48	-21 4)/2
g_9	(0	1	-4	21	-44	64	-67	48	-21 4)/2

Family F18:3d (transpose of F18:3c). Largest known from $k = 21$ for $k \equiv 21 \pmod{2}$.
 odd-girth maximum from $k = 21$. Maximal levels: $(2k + 12)/9$ from $k = 12$.

LGM odd basis									
	$2a - 1$	1	1	0	0	0	1	1	1
	-1	$2a - 1$	1	0	1	1	1	1	1
	-1	-1	$2a - 1$	1	1	1	0	0	0
	0	0	-1	$2a - 1$	-1	-1	0	-1	-1
	0	-1	-1	1	$2a$	0	-1	0	0
	0	-1	-1	1	0	$2a$	-1	-1	-1
	-1	-1	0	0	1	1	$2a$	0	0
	-1	-1	0	1	0	1	0	$2a$	0
	-1	-1	0	1	0	1	0	0	$2a$
Polynomial in $2a$									
Order	(1	-4	28	-70	118	-132	96	-42	8 0)/2
g_1	(0	0	2	-8	20	-34	29	-18	6 0)/2
g_2	(0	0	2	-1	0	8	-9	6	-2 0)/2
g_3	(0	0	0	8	-19	31	-30	12	-2 0)/2
g_4	(0	0	1	0	-12	19	-20	18	-6 0)/2
g_5	(0	0	1	-7	14	-12	-2	7	-2 0)/2
g_6	(0	0	2	-11	28	-42	44	-27	6 0)/2
g_7	(0	1	-4	21	-43	56	-52	34	-10 0)/2
g_8	(0	0	0	7	-20	36	-49	44	-21 4)/2
g_9	(0	1	-4	21	-50	82	-83	52	-21 4)/2

Table A.57: Degree 18, diameter class $k \equiv 4 \pmod{9}$, $a = (2k + 1)/9$

LGM odd basis	Polynomial in $2a$
Family F18:4 (self-transpose, self-conjugate, translate of F19:0). Graphs are largest known from $k = 4$. odd-girth maximum from $k = 13$. Maximal levels: $(2k + 10)/9$ from $k = 4$.	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 & 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 2a & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2a & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2a & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 & 2a & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 & 2a & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 2a \end{pmatrix}$	Order (1 0 20 0 58 0 43 0 9 0) /2 g_1 (0 1 0 16 0 30 0 13 0 1) g_2 (0 0 1 0 7 -8 9 -7 2 -1) g_3 (0 0 1 2 5 8 9 5 4 1) g_4 (0 0 1 -1 9 6 10 6 2 1) g_5 (0 0 1 -1 7 -6 2 -4 0 -1) g_6 (0 0 0 1 -10 1 -15 3 -5 1) g_7 (0 0 0 4 -1 11 0 3 0 -1) g_8 (0 0 0 3 2 9 -5 5 -3 1) g_9 (0 0 0 2 7 1 12 3 4 1)

Note: generator polynomials not divided by 2

Table A.58: Degree 18, diameter class $k \equiv 5 \pmod{9}$, except $k \equiv 5 \pmod{27}$,
 $a = (2k - 1)/9$

LGM odd basis	Polynomial in $2a$
Family F18:5a (transpose of F18:5b, conjugate of F18:3a). Graphs are largest known from $k = 14$. odd-girth maximum from $k = 14$. Maximal levels: $(2k + 8)/9$ from $k = 14$.	
$\begin{pmatrix} 2a+1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a+1 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 2a+1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 2a & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2a & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 2a & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 2a & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 2a \end{pmatrix}$	Order (1 4 29 74 115 122 81 34 8 0)/2 g_1 (0 1 6 34 69 88 67 32 10 0)/2 g_2 (0 0 2 -6 -8 -15 -20 -7 -2 0)/2 g_3 (0 0 0 8 39 48 17 -2 -2 0)/2 g_4 (0 0 3 8 21 21 12 6 2 0)/2 g_5 (0 0 1 4 -15 -59 -78 -51 -14 0)/2 g_6 (0 0 3 11 37 81 81 35 6 0)/2 g_7 (0 0 1 15 43 65 68 45 17 4)/2 g_8 (0 1 3 14 31 50 54 36 17 4)/2 g_9 (0 2 7 44 101 111 69 22 2 0)/2
Family F18:5b (transpose of F18:5a, conjugate of F18:3b). Graphs are largest known from $k = 14$. odd-girth maximum from $k = 14$. Maximal levels: $(2k + 8)/9$ from $k = 14$.	
$\begin{pmatrix} 2a+1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 2a+1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 2a+1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 2a+1 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 2a & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2a & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 2a & 0 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 & 0 & 2a & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 2a \end{pmatrix}$	Order (1 4 29 74 115 122 81 34 8 0)/2 g_1 (0 1 4 22 39 48 37 16 4 0)/2 g_2 (0 0 2 6 8 7 4 1 0 0)/2 g_3 (0 0 2 6 17 28 25 14 4 0)/2 g_4 (0 0 1 -4 -9 -19 -18 -10 -4 0)/2 g_5 (0 0 1 -2 -5 -5 -6 -3 0 0)/2 g_6 (0 0 1 -1 -7 -11 -9 -3 0 0)/2 g_7 (0 0 1 11 27 43 48 33 15 4)/2 g_8 (0 1 3 18 47 72 74 48 19 4)/2 g_9 (0 0 1 8 23 31 25 14 4 0)/2

Table A.59: Degree 18, diameter class $k \equiv 5 \pmod{9}$, $a = (2k - 1)/9$

LGM odd basis										Polynomial in $2a$									
Family F18:5c (transpose of F18:5d). Largest known from $k = 32$ for $k \equiv 5 \pmod{2}7$. odd-girth maximum from $k = 14$. Maximal levels: $(2k + 8)/9$ from $k = 14$.																			
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 2a+1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2a+1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 2a & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 2a & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 2a & 0 \end{pmatrix}$	Order	(1	4	28	72	117	122	89	42	8	0)/2							
	g_1	(0	2	10	44	80	101	77	43	8	0)/2							
	g_2	(0	0	2	-10	-31	-51	-49	-20	-4	0)/2							
	g_3	(0	0	1	15	41	55	51	34	12	0)/2							
	g_4	(0	0	1	21	43	71	60	44	12	0)/2							
	g_5	(0	0	5	20	44	61	65	46	23	4)/2							
	g_6	(0	1	-1	8	28	56	57	43	19	4)/2							
	g_7	(0	0	4	18	49	76	68	30	4	0)/2							
	g_8	(0	1	3	10	26	52	67	53	16	0)/2							
	g_9	(0	2	7	36	70	89	62	25	8	0)/2							
Family F18:5d (transpose of F18:5c). Largest known from $k = 32$ for $k \equiv 5 \pmod{2}7$. odd-girth maximum from $k = 14$. Maximal levels: $(2k + 8)/9$ from $k = 14$.																			
$\begin{pmatrix} 2a+1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2a+1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 2a+1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2a+1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 2a & 0 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & 2a & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & 2a & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 2a & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2a & 0 \end{pmatrix}$	Order	(1	4	28	72	117	122	89	42	8	0)/2							
	g_1	(0	0	2	0	-5	-9	-6	-5	-2	0)/2							
	g_2	(0	0	2	3	7	10	11	8	2	0)/2							
	g_3	(0	0	1	5	3	-8	-14	-8	-2	0)/2							
	g_4	(0	0	2	10	21	30	24	12	2	0)/2							
	g_5	(0	0	0	10	36	65	69	47	21	4)/2							
	g_6	(0	1	4	18	36	52	53	42	21	4)/2							
	g_7	(0	1	4	20	49	66	47	16	2	0)/2							
	g_8	(0	0	1	8	11	2	-12	-11	-2	0)/2							
	g_9	(0	0	2	10	28	44	44	25	6	0)/2							

Table A.60: Degree 18, diameter class $k \equiv 6 \pmod{9}$, $a = (2k - 3)/9$

Family F18:6a (transpose of F18:6b, conjugate of F18:2a, translate of F19:2a).

Graphs are largest known from $k = 6$. odd-girth maximum from $k = 15$.Maximal levels: $(2k + 6)/9$ from $k = 6$.

LGM odd basis										
	$2a$	-1	-1	-1	-1	0	0	0	0	
	1	$2a$	0	0	-1	-1	-1	-1	-1	
	1	0	$2a + 1$	0	-1	-1	-1	-1	0	
	1	0	0	$2a + 1$	-1	-1	-1	0	-1	
	1	1	1	1	$2a + 1$	-1	-1	0	-1	
	0	1	1	1	1	$2a + 1$	0	1	0	
	0	1	1	1	1	0	$2a + 1$	1	0	
	0	1	1	0	0	-1	-1	$2a + 1$	-1	
	0	1	0	1	1	0	0	1	$2a + 2$	
Polynomial in $2a$										
Order	(1	8	50	194	462	698	672	394	125	16)/2
g_1	(0	2	16	104	395	857	1094	817	329	54)/2
g_2	(0	0	2	11	27	39	52	67	50	14)/2
g_3	(0	0	0	0	37	123	171	129	54	10)/2
g_4	(0	0	1	0	-27	-144	-320	-346	-178	-34)/2
g_5	(0	0	1	-4	-30	-74	-88	-51	-14	-2)/2
g_6	(0	0	1	13	48	95	119	87	28	2)/2
g_7	(0	1	6	30	103	216	268	198	81	14)/2
g_8	(0	1	10	54	210	516	733	574	226	34)/2
g_9	(0	2	11	64	199	346	367	233	78	10)/2

Family F18:6b (transpose of F18:6a, conjugate of F18:2b, translate of F19:2b)

Graphs are largest known from $k = 6$. odd-girth maximum from $k = 15$.Maximal levels: $(2k + 6)/9$ from $k = 6$.

LGM odd basis										
	$2a$	1	1	1	1	0	0	0	0	
	-1	$2a$	0	0	1	1	1	1	1	
	-1	0	$2a + 1$	0	1	1	1	1	0	
	-1	0	0	$2a + 1$	1	1	1	0	1	
	-1	-1	-1	-1	$2a + 1$	1	1	0	1	
	0	-1	-1	-1	-1	$2a + 1$	0	-1	0	
	0	-1	-1	-1	-1	0	$2a + 1$	-1	0	
	0	-1	-1	0	0	1	1	$2a + 1$	1	
	0	-1	0	-1	-1	0	0	-1	$2a + 2$	
Polynomial in $2a$										
Order	(1	8	50	194	462	698	672	394	125	16)/2
g_1	(0	0	0	7	36	79	96	68	26	4)/2
g_2	(0	0	2	9	18	15	-4	-19	-15	-4)/2
g_3	(0	0	1	2	-14	-57	-92	-75	-29	-4)/2
g_4	(0	0	2	9	23	44	59	49	22	4)/2
g_5	(0	0	2	16	52	94	105	71	26	4)/2
g_6	(0	0	0	9	41	90	113	82	30	4)/2
g_7	(0	1	7	34	110	221	274	203	79	12)/2
g_8	(0	0	2	14	52	107	127	86	30	4)/2
g_9	(0	1	6	30	86	145	148	89	29	4)/2

Table A.61: Degree 18, diameter class $k \equiv 7 \pmod{9}$, $a = (2k + 4)/9$

Family F18:7a (transpose of F18:7b, conjugate of F18:1c). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 16$. Maximal levels: $(2k + 4)/9$ from $k = 7$.										
LGM odd basis										
	$2a$	-1	-1	-1	-1	0	0	0	0	
	1	$2a$	0	0	-1	-1	-1	0	0	
	1	0	$2a$	-1	-1	-1	-1	-1	0	
	1	0	1	$2a - 1$	0	-1	-1	0	-1	
	1	1	1	0	$2a - 1$	-1	-1	-1	-1	
	0	1	1	1	1	$2a - 1$	0	0	-1	
	0	1	1	1	1	0	$2a - 1$	0	-1	
	0	0	1	0	1	0	0	$2a - 1$	-1	
	0	0	0	1	1	1	1	1	$2a - 1$	
Polynomial in $2a$										
Order	$(1$	-6	37	-122	251	-342	305	-172	56	$-8)/2$
g_1	$(0$	2	-12	76	-221	358	-345	183	-43	$2)/2$
g_2	$(0$	0	1	-13	69	-177	231	-156	51	$-6)/2$
g_3	$(0$	0	0	11	-49	112	-166	154	-78	$16)/2$
g_4	$(0$	0	1	0	-16	12	34	-53	26	$-4)/2$
g_5	$(0$	0	2	-1	-10	38	-64	44	-7	$-2)/2$
g_6	$(0$	0	2	-12	28	-36	28	-18	9	$-2)/2$
g_7	$(0$	1	-3	20	-62	125	-153	106	-39	$6)/2$
g_8	$(0$	1	-3	20	-49	53	-36	26	-16	$4)/2$
g_9	$(0$	2	-12	63	-208	403	-473	328	-121	$18)/2$
Family F18:7b (transpose of F18:7a, conjugate of F18:1d). Graphs are largest known from $k = 7$. odd-girth maximum from $k = 16$. Maximal levels: $(2k + 4)/9$ from $k = 7$.										
LGM odd basis										
	$2a$	1	1	1	1	0	0	0	0	
	-1	$2a$	0	0	1	1	1	0	0	
	-1	0	$2a$	1	1	1	1	1	0	
	-1	0	-1	$2a - 1$	0	1	1	0	1	
	-1	-1	-1	0	$2a - 1$	1	1	1	1	
	0	-1	-1	-1	-1	$2a - 1$	0	0	1	
	0	-1	-1	-1	-1	0	$2a - 1$	0	1	
	0	0	-1	0	-1	0	0	$2a - 1$	1	
	0	0	0	-1	-1	-1	-1	-1	$2a - 1$	
Polynomial in $2a$										
Order	$(1$	-6	37	-122	251	-342	305	-172	56	$-8)/2$
g_1	$(0$	2	-12	76	-265	522	-627	449	-171	$26)/2$
g_2	$(0$	0	1	-1	25	-101	163	-128	47	$-6)/2$
g_3	$(0$	0	0	7	-15	14	6	-40	40	$-12)/2$
g_4	$(0$	0	1	-4	0	40	-90	79	-30	$4)/2$
g_5	$(0$	0	2	-11	30	-70	98	-74	31	$-6)/2$
g_6	$(0$	0	2	-4	22	-64	110	-102	43	$-6)/2$
g_7	$(0$	1	-7	36	-112	225	-291	226	-91	$14)/2$
g_8	$(0$	1	-7	36	-115	197	-178	86	-24	$4)/2$
g_9	$(0$	2	-8	47	-106	117	-51	-28	37	$-10)/2$

Table A.62: Degree 18, diameter class $k \equiv 8 \pmod{9}$, $a = (2k + 2)/9$

LGM odd basis	Polynomial in $2a$
Family F18:8a (transpose of F18:8b, conjugate of F18:0a). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 17$. Maximal levels: $(2k + 11)/9$ from $k = 8$.	
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 2a-1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 2a & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2a & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order (1 -2 23 -34 59 -52 35 -16 4 0)/2 g_1 (0 1 -2 15 -15 24 -15 7 -2 0)/2 g_2 (0 0 2 -1 6 -4 2 -4 0 0)/2 g_3 (0 0 2 -4 5 -2 -2 5 -2 0)/2 g_4 (0 0 2 -2 11 -12 15 -7 2 0)/2 g_5 (0 0 1 -10 16 -27 25 -18 9 -2)/2 g_6 (0 1 -1 13 -18 32 -27 17 -7 2)/2 g_7 (0 0 0 7 -5 14 -10 4 0 0)/2 g_8 (0 0 1 4 -2 9 -1 -2 0 0)/2 g_9 (0 0 1 2 -8 8 -11 6 0 0)/2
Family F18:8b (transpose of F18:8a, conjugate of F18:0b). Graphs are largest known from $k = 8$. odd-girth maximum from $k = 17$. Maximal levels: $(2k + 11)/9$ from $k = 8$.	
$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2a-1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 2a & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2a & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 2a & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 2a & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & 2a & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 2a \end{pmatrix}$	Order (1 -2 23 -34 59 -52 35 -16 4 0)/2 g_1 (0 1 0 13 -7 18 -7 3 0 0)/2 g_2 (0 0 2 -1 4 2 -2 2 -2 0)/2 g_3 (0 0 2 -2 -1 0 -8 3 0 0)/2 g_4 (0 0 2 -4 13 -12 17 -11 4 0)/2 g_5 (0 0 1 8 -8 21 -17 14 -7 2)/2 g_6 (0 1 -3 15 -26 38 -35 21 -9 2)/2 g_7 (0 0 0 7 -13 18 -14 6 -2 0)/2 g_8 (0 0 1 -8 10 -21 15 -8 2 0)/2 g_9 (0 0 1 -6 2 -4 -1 4 -2 0)/2

Table A.63: Degree 19, diameter class $k \equiv 0 \pmod{9}$, $a = 2k/9$

LGM	Polynomial in $2a$
Family F19:0 (self-transpose, self-conjugate, translate of F18:4). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 18$. Maximal levels: $(2k + 9)/9$ from $k = 9$.	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2a & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 & 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 2a & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2a & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2a & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 & 2a & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 & 2a & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 2a \end{pmatrix}$	Order (1 0 20 0 58 0 43 0 9 0) g_1 (0 1 4 15 45 30 57 13 17 1) g_2 (0 1 -2 11 -25 22 -23 9 -3 1) g_3 (0 1 0 8 1 18 -6 8 -3 -1) g_4 (0 1 -1 14 9 9 3 -4 -1 -1) g_5 (0 1 -2 12 -13 8 -4 4 -1 1) g_6 (0 1 0 26 -4 40 -16 17 -6 1) g_7 (0 0 2 -6 -1 -11 -20 -3 -10 1) g_8 (0 0 2 -4 5 -33 8 -17 4 -1) g_9 (0 1 1 20 -13 37 -11 10 -3 -1)

Table A.64: Degree 19, diameter class $k \equiv 1 \pmod{9}$, $a = (2k - 2)/9$

LGM	Polynomial in $2a$										
Family F19:1a (transpose of F19:1b). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 19$. Maximal levels: $(2k + 7)/9$ from $k = 10$.											
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 2a+1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 2a+1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2a & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 2a & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 & 2a & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 2a & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 2a \end{pmatrix}$	Order	(1	4	27	68	122	146	119	66	23	4)
	g_1	(0	1	8	30	64	97	92	60	25	5)
	g_2	(0	1	7	27	59	64	49	26	10	3)
	g_3	(0	1	4	11	23	34	34	17	7	1)
	g_4	(0	1	2	10	5	2	1	2	-4	-3)
	g_5	(0	1	0	4	8	18	25	18	5	1)
	g_6	(0	1	1	6	5	-17	-43	-42	-18	-3)
	g_7	(0	0	0	13	40	70	61	25	0	-3)
	g_8	(0	1	2	13	45	82	91	68	33	7)
	g_9	(0	0	2	-5	-7	-6	-14	-19	-16	-5)
Family F19:1b (transpose of F19:1a). Graphs are largest known from $k = 10$. odd-girth maximum from $k = 19$. Maximal levels: $(2k + 7)/9$ from $k = 10$.											
$\begin{pmatrix} 2a+1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 2a+1 & 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2a+1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 2a & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 & 2a & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 0 & 2a & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 & 2a & -1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2a \end{pmatrix}$	Order	(1	4	27	68	122	146	119	66	23	4)
	g_1	(0	1	-2	-7	-12	-10	-4	3	0	1)
	g_2	(0	1	-2	-3	-24	-48	-54	-41	-16	-3)
	g_3	(0	1	4	1	0	-14	-20	-21	-12	-3)
	g_4	(0	1	4	9	8	-10	-32	-33	-18	-5)
	g_5	(0	1	10	41	88	130	126	79	30	7)
	g_6	(0	1	8	31	66	90	84	55	20	3)
	g_7	(0	1	6	13	32	52	56	43	22	5)
	g_8	(0	1	0	15	54	98	108	73	28	5)
	g_9	(0	1	4	7	4	10	14	13	8	1)

Table A.65: Degree 19, diameter class $k \equiv 2 \pmod{9}$, $a = (2k - 4)/9$

Family F19:2a (transpose of F19:2b, conjugate of F19:7a, translate of F18:6a). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 20$. Maximal levels: $(2k + 5)/9$ from $k = 11$.										
LGM										
	$2a$	-1	-1	-1	-1	0	0	0	0	
	1	$2a$	0	0	-1	-1	-1	-1	-1	
	1	0	$2a + 1$	0	-1	-1	-1	-1	0	
	1	0	0	$2a + 1$	-1	-1	-1	0	-1	
	1	1	1	1	$2a + 1$	-1	-1	0	-1	
	0	1	1	1	1	$2a + 1$	0	1	0	
	0	1	1	1	1	0	$2a + 1$	1	0	
	0	1	1	0	0	-1	-1	$2a + 1$	-1	
	0	1	0	1	1	0	0	1	$2a + 2$	
Polynomial in $2a$										
Order	(1	8	50	194	462	698	672	394	125	16)
g_1	(0	1	12	81	298	639	824	626	254	41)
g_2	(0	1	10	41	90	115	88	45	19	5)
g_3	(0	1	9	36	74	87	70	41	13	1)
g_4	(0	1	8	27	45	12	-89	-149	-92	-19)
g_5	(0	1	4	0	-32	-88	-115	-77	-24	-3)
g_6	(0	1	2	11	61	166	241	194	80	13)
g_7	(0	0	5	32	90	145	146	91	29	3)
g_8	(0	1	6	18	58	165	283	260	116	19)
g_9	(0	0	4	20	46	69	66	35	9	1)
Family F19:2b (transpose of F19:2a, conjugate of F19:7b, translate of F18:6b). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 20$. Maximal levels: $(2k + 5)/9$ from $k = 11$.										
LGM										
	$2a$	1	1	1	1	0	0	0	0	
	-1	$2a$	0	0	1	1	1	1	1	
	-1	0	$2a + 1$	0	1	1	1	1	0	
	-1	0	0	$2a + 1$	1	1	1	0	1	
	-1	-1	-1	-1	$2a + 1$	1	1	0	1	
	0	-1	-1	-1	-1	$2a + 1$	0	-1	0	
	0	-1	-1	-1	-1	0	$2a + 1$	-1	0	
	0	-1	-1	0	0	1	1	$2a + 1$	1	
	0	-1	0	-1	-1	0	0	-1	$2a + 2$	
Polynomial in $2a$										
Order	(1	8	50	194	462	698	672	394	125	16)
g_1	(0	1	4	23	108	281	410	336	142	23)
g_2	(0	1	6	15	20	19	26	41	35	11)
g_3	(0	1	5	14	28	51	74	63	25	3)
g_4	(0	1	6	17	35	50	33	-9	-22	-7)
g_5	(0	1	10	40	98	168	203	157	64	9)
g_6	(0	1	12	73	239	470	573	418	162	25)
g_7	(0	0	5	30	88	159	186	133	53	9)
g_8	(0	1	8	32	66	57	-17	-66	-40	-7)
g_9	(0	0	4	22	60	99	96	53	17	3)

Table A.66: Degree 19, diameter class $k \equiv 3 \pmod{9}$, $a = (2k + 3)/9$

Family F19:3a (transpose of F19:3b). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 21$. Maximal levels: $(2k + 3)/9$ from $k = 2$.										
LGM										
	$2a$	0	0	0	0	-1	-1	-1	-1	
	0	$2a$	0	0	-1	-1	-1	-1	0	
	0	0	$2a$	-1	-1	-1	-1	-1	-1	
	0	0	1	$2a - 1$	-1	-1	0	0	0	
	0	1	1	1	$2a - 1$	0	1	1	0	
	1	1	1	1	0	$2a - 1$	1	1	1	
	1	1	1	0	-1	-1	$2a - 1$	0	0	
	1	1	1	0	-1	-1	0	$2a - 1$	0	
	1	0	1	0	0	-1	0	0	$2a - 1$	
Polynomial in $2a$										
Order	(1	-6	36	-118	245	-338	313	-190	67	-10)
g_1	(0	1	-6	30	-105	214	-265	199	-81	13)
g_2	(0	1	-5	24	-67	107	-105	54	-8	-1)
g_3	(0	1	-4	19	-45	72	-80	65	-37	9)
g_4	(0	1	-4	21	-32	13	30	-48	22	-3)
g_5	(0	1	-7	39	-110	172	-164	92	-26	3)
g_6	(0	1	-8	45	-135	255	-307	233	-103	19)
g_7	(0	0	1	1	-10	18	-16	5	-2	1)
g_8	(0	1	-4	32	-97	176	-196	138	-59	11)
g_9	(0	0	1	-1	3	-12	19	-19	12	-3)
Family F19:3b (transpose of F19:3a). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 21$. Maximal levels: $(2k + 3)/9$ from $k = 2$.										
LGM										
	$2a$	0	0	0	0	1	1	1	1	
	0	$2a$	0	0	1	1	1	1	0	
	0	0	$2a$	1	1	1	1	1	1	
	0	0	-1	$2a - 1$	1	1	0	0	0	
	0	-1	-1	-1	$2a - 1$	0	-1	-1	0	
	-1	-1	-1	-1	0	$2a - 1$	-1	-1	-1	
	-1	-1	-1	0	1	1	$2a - 1$	0	0	
	-1	-1	-1	0	1	1	0	$2a - 1$	0	
	-1	0	-1	0	0	1	0	0	$2a - 1$	
Polynomial in $2a$										
Order	(1	-6	36	-118	245	-338	313	-190	67	-10)
g_1	(0	1	-8	38	-83	96	-53	-3	19	-7)
g_2	(0	1	-9	40	-107	185	-209	148	-60	11)
g_3	(0	1	-10	45	-111	180	-196	145	-69	15)
g_4	(0	1	-6	21	-60	107	-120	80	-24	1)
g_5	(0	1	-1	7	-24	36	-28	8	2	-1)
g_6	(0	1	0	1	-9	33	-55	51	-29	7)
g_7	(0	1	-4	6	3	-20	34	-28	13	-3)
g_8	(0	0	1	-25	90	-178	214	-161	70	-13)
g_9	(0	0	1	-17	65	-120	123	-71	20	-1)

Table A.67: Degree 19, diameter class $k \equiv 4 \pmod{9}$, $a = (2k + 1)/9$

Family F19:4 (self-transpose, conjugate of F19:5). Graphs are largest known from $k = 13$. odd-girth maximum from $k = 13$. Maximal levels: $(2k + 10)/9$ from $k = 13$.

LGM										
	$2a - 1$	-1	-1	-1	-1	-1	0	0	0	
	1	$2a - 1$	0	0	0	-1	-1	-1	-1	
	1	0	$2a$	-1	0	-1	-1	-1	-1	
	1	0	1	$2a$	0	0	-1	-1	0	
	1	0	0	0	$2a$	0	-1	-1	-1	
	1	1	1	0	0	$2a$	-1	0	0	
	0	1	1	1	1	1	$2a$	0	1	
	0	1	1	1	1	0	0	$2a$	0	
	0	1	1	0	1	0	-1	0	$2a$	

Polynomial in $2a$										
Order	$(1$	-2	22	-32	62	-60	47	-26	9	$-2)$
g_1	$(0$	1	-6	32	-34	66	-42	31	-12	$3)$
g_2	$(0$	1	-4	6	-12	8	-6	-1	-2	$-1)$
g_3	$(0$	1	-6	12	-22	20	-14	5	2	$-1)$
g_4	$(0$	1	-2	4	0	0	0	-3	2	$1)$
g_5	$(0$	1	-4	10	-16	32	-26	25	-10	$3)$
g_6	$(0$	1	0	2	12	-12	20	-9	2	$-1)$
g_7	$(0$	1	4	6	6	14	-6	7	-2	$-1)$
g_8	$(0$	1	2	6	0	12	0	1	0	$1)$
g_9	$(0$	1	0	4	-10	6	-14	7	-4	$1)$

Table A.68: Degree 19, diameter class $k \equiv 5 \pmod{9}$, $a = (2k - 1)/9$

LGM										
Polynomial in $2a$										

Family F19:5 (self-transpose, conjugate of F19:4). Graphs are largest known from $k = 14$. odd-girth maximum from $k = 14$. Maximal levels: $(2k + 8)/9$ from $k = 5$.

	1	$2a + 1$	0	0	0	-1	-1	-1	-1	
	1	0	$2a$	-1	0	-1	-1	-1	-1	
	1	0	1	$2a$	0	0	-1	-1	0	
	1	0	0	0	$2a$	0	-1	-1	-1	
	1	1	1	0	0	$2a$	-1	0	0	
	0	1	1	1	1	1	$2a$	0	1	
	0	1	1	1	1	0	0	$2a$	0	
	0	1	1	0	1	0	-1	0	$2a$	

Order	$(1$	2	22	32	62	60	47	26	9	$2)$
g_1	$(0$	1	6	32	34	66	42	31	12	$3)$
g_2	$(0$	1	4	6	12	8	6	-1	2	$-1)$
g_3	$(0$	1	6	12	22	20	14	5	-2	$-1)$
g_4	$(0$	1	2	4	0	0	0	-3	-2	$1)$
g_5	$(0$	1	4	10	16	32	26	25	10	$3)$
g_6	$(0$	1	0	2	-12	-12	-20	-9	-2	$-1)$
g_7	$(0$	1	-4	6	-6	14	6	7	2	$-1)$
g_8	$(0$	1	-2	6	0	12	0	1	0	$1)$
g_9	$(0$	1	0	4	10	6	14	7	4	$1)$

Table A.69: Degree 19, diameter class $k \equiv 6 \pmod{9}$, $a = (2k - 3)/9$

Family F19:6a (transpose of F19:6b). Graphs are largest known from $k = 6$.
odd-girth maximum from $k = 15$. Maximal levels: $(2k + 6)/9$ from $k = 6$.

LGM										
	$2a$	0	-1	-1	-1	-1	-1	-1	-1	0
	0	$2a$	-1	-1	-1	-1	0	0	0	0
	1	1	$2a$	-1	-1	0	-1	0	-1	-1
	1	1	1	$2a - 1$	0	0	0	1	-1	-1
	1	1	1	0	$2a - 1$	0	0	1	-1	-1
	1	1	0	0	0	$2a - 1$	0	0	-1	-1
	1	0	1	0	0	0	$2a - 1$	1	0	0
	1	0	0	-1	-1	0	-1	$2a - 1$	-1	-1
	0	0	1	1	1	1	0	1	$2a - 1$	-1

Polynomial in $2a$										
Order	(1	6	36	120	253	350	317	184	63	10)
g_1	(0	1	10	48	123	199	209	141	64	17)
g_2	(0	1	9	45	125	226	260	176	61	7)
g_3	(0	1	6	14	5	-42	-105	-117	-67	-17)
g_4	(0	1	2	4	12	33	53	50	28	9)
g_5	(0	0	3	27	77	131	133	81	25	1)
g_6	(0	1	4	14	33	50	44	25	8	-1)
g_7	(0	0	3	19	60	101	106	69	27	7)
g_8	(0	1	6	17	33	50	71	72	37	7)
g_9	(0	1	1	20	78	169	221	166	69	13)

Family F19:6b (transpose of F19:6a). Graphs are largest known from $k = 6$.
odd-girth maximum from $k = 15$. Maximal levels: $(2k + 6)/9$ from $k = 6$.

LGM										
	$2a$	0	1	1	1	1	1	1	1	0
	0	$2a$	1	1	1	1	0	0	0	0
	-1	-1	$2a$	1	1	0	1	0	1	1
	-1	-1	-1	$2a - 1$	0	0	0	-1	1	1
	-1	-1	-1	0	$2a - 1$	0	0	-1	1	1
	-1	-1	0	0	0	$2a - 1$	0	0	1	1
	-1	0	-1	0	0	0	$2a - 1$	-1	0	0
	-1	0	0	1	1	0	1	$2a - 1$	1	1
	0	0	-1	-1	-1	-1	0	-1	$2a - 1$	-1

Polynomial in $2a$										
Order	(1	6	36	120	253	350	317	184	63	10)
g_1	(0	1	2	-4	-27	-59	-73	-55	-24	-5)
g_2	(0	1	3	5	1	-12	-20	-14	-5	-1)
g_3	(0	1	8	24	45	58	51	31	13	3)
g_4	(0	1	10	46	112	173	171	108	40	7)
g_5	(0	0	5	15	23	9	-15	-23	-13	-3)
g_6	(0	1	8	34	85	130	124	73	24	3)
g_7	(0	0	3	13	22	21	10	-1	-3	-1)
g_8	(0	1	6	15	25	32	31	20	7	1)
g_9	(0	1	1	-2	-12	-25	-27	-18	-7	-1)

Table A.70: Degree 19, diameter class $k \equiv 7 \pmod{9}$, $a = (2k + 4)/9$

Family F19:7a (transpose of F19:7b, conjugate of F19:2a, translate of F18:2a). Graphs are largest known from $k = 16$. odd-girth maximum from $k = 16$. Maximal levels: $(2k + 4)/9$ from $k = 7$.										
LGM										
	$2a$	-1	-1	-1	-1	0	0	0	0	
	1	$2a$	0	0	-1	-1	-1	-1	-1	
	1	0	$2a - 1$	0	-1	-1	-1	-1	0	
	1	0	0	$2a - 1$	-1	-1	-1	0	-1	
	1	1	1	1	$2a - 1$	-1	-1	0	-1	
	0	1	1	1	1	$2a - 1$	0	1	0	
	0	1	1	1	1	0	$2a - 1$	1	0	
	0	1	1	0	0	-1	-1	$2a - 1$	-1	
	0	1	0	1	1	0	0	1	$2a - 2$	
Polynomial in $2a$										
Order	(1	-8	50	-194	462	-698	672	-394	125	-16)
g_1	(0	1	-4	23	-108	281	-410	336	-142	23)
g_2	(0	1	-6	15	-20	19	-26	41	-35	11)
g_3	(0	1	-5	14	-28	51	-74	63	-25	3)
g_4	(0	1	-6	17	-35	50	-33	-9	22	-7)
g_5	(0	1	-10	40	-98	168	-203	157	-64	9)
g_6	(0	1	-12	73	-239	470	-573	418	-162	25)
g_7	(0	0	5	-30	88	-159	186	-133	53	-9)
g_8	(0	1	-8	32	-66	57	17	-66	40	-7)
g_9	(0	0	4	-22	60	-99	96	-53	17	-3)
Family F19:7b (transpose of F19:7a, conjugate of F19:2b, translate of F18:2b). Graphs are largest known from $k = 16$. odd-girth maximum from $k = 16$. Maximal levels: $(2k + 4)/9$ from $k = 7$.										
LGM										
	$2a$	1	1	1	1	0	0	0	0	
	-1	$2a$	0	0	1	1	1	1	1	
	-1	0	$2a - 1$	0	1	1	1	1	0	
	-1	0	0	$2a - 1$	1	1	1	0	1	
	-1	-1	-1	-1	$2a - 1$	1	1	0	1	
	0	-1	-1	-1	-1	$2a - 1$	0	-1	0	
	0	-1	-1	-1	-1	0	$2a - 1$	-1	0	
	0	-1	-1	0	0	1	1	$2a - 1$	1	
	0	-1	0	-1	-1	0	0	-1	$2a - 2$	
Polynomial in $2a$										
Order	(1	-8	50	-194	462	-698	672	-394	125	-16)
g_1	(0	1	-12	81	-298	639	-824	626	-254	41)
g_2	(0	1	-10	41	-90	115	-88	45	-19	5)
g_3	(0	1	-9	36	-74	87	-70	41	-13	1)
g_4	(0	1	-8	27	-45	12	89	-149	92	-19)
g_5	(0	1	-4	0	32	-88	115	-77	24	-3)
g_6	(0	1	-2	11	-61	166	-241	194	-80	13)
g_7	(0	0	5	-32	90	-145	146	-91	29	-3)
g_8	(0	1	-6	18	-58	165	-283	260	-116	19)
g_9	(0	0	4	-20	46	-69	66	-35	9	-1)

Table A.71: Degree 19, diameter class $k \equiv 8 \pmod{9}$, $a = (2k + 2)/9$

Family F19:8a (transpose of F19:8b). Graphs are largest known from $k = 17$. odd-girth maximum from $k = 17$. Maximal levels: $(2k + 2)/9$ from $k = 8$.										
LGM										
	$2a - 1$	-1	-1	-1	0	0	0	-1	-1	
	1	$2a - 1$	0	-1	-1	-1	-1	-1	-1	
	1	0	$2a - 1$	-1	-1	-1	0	-1	0	
	1	1	1	$2a - 1$	-1	-1	0	0	0	
	0	1	1	1	$2a$	0	0	0	1	
	0	1	1	1	0	$2a$	1	1	1	
	0	1	0	0	0	-1	$2a$	0	0	
	1	1	1	0	0	-1	0	$2a$	0	
	1	1	0	0	-1	-1	0	0	$2a$	

Polynomial in $2a$										
Order	(1	-4	27	-66	109	-126	104	-62	23	-4)
g_1	(0	1	2	17	-42	69	-72	50	-22	3)
g_2	(0	1	2	-9	18	-13	8	0	-6	1)
g_3	(0	1	0	-3	4	-15	18	-16	12	-3)
g_4	(0	1	-4	1	6	-13	20	-14	8	-3)
g_5	(0	1	-8	33	-60	79	-70	40	-16	3)
g_6	(0	1	-10	43	-88	121	-114	78	-34	5)
g_7	(0	1	-4	15	-14	3	8	-18	16	-5)
g_8	(0	1	-6	13	-28	41	-46	36	-18	5)
g_9	(0	1	-4	5	2	-15	18	-16	6	1)

Family F19:8b (transpose of F19:8a). Graphs are largest known from $k = 17$. odd-girth maximum from $k = 17$. Maximal levels: $(2k + 2)/9$ from $k = 8$.										
LGM										
	$2a - 1$	1	1	1	0	0	0	1	1	
	-1	$2a - 1$	0	1	1	1	1	1	1	
	-1	0	$2a - 1$	1	1	1	0	1	0	
	-1	-1	-1	$2a - 1$	1	1	0	0	0	
	0	-1	-1	-1	$2a$	0	0	0	-1	
	0	-1	-1	-1	0	$2a$	-1	-1	-1	
	0	-1	0	0	0	1	$2a$	0	0	
	-1	-1	-1	0	0	1	0	$2a$	0	
	-1	-1	0	0	1	1	0	0	$2a$	

Polynomial in $2a$										
Order	(1	-4	27	-66	109	-126	104	-62	23	-4)
g_1	(0	1	-8	41	-74	97	-84	50	-18	1)
g_2	(0	1	-8	21	-42	57	-56	40	-20	5)
g_3	(0	1	-6	15	-22	19	-14	2	4	-1)
g_4	(0	1	-2	1	4	-5	4	0	-2	1)
g_5	(0	1	0	7	-20	37	-42	28	-10	1)
g_6	(0	1	2	1	0	1	-4	4	-4	1)
g_7	(0	1	-4	17	-46	67	-70	52	-26	7)
g_8	(0	1	-2	3	10	-29	46	-46	26	-7)
g_9	(0	1	-4	9	-12	13	-10	10	-4	-1)

A.9 Circulant graph families of degree 20

Table A.72: Degree 20, diameter class $k \equiv 0 \pmod{5}$, $a = k/5$

Family F20:0a (transpose of F20:0b). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 5$. Maximal levels: $(k + 5)/5$ from $k = 10$.											
LGM odd basis											
$\begin{pmatrix} 2a + 1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2a + 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 2a & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 2a & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2a & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2a & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 2a & 0 \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	2	26	42	93	92	86	46	16	4	0)/2
g_1	(0	1	0	17	11	42	19	26	4	2	0)/2
g_2	(0	0	1	3	3	5	6	2	4	-2	0)/2
g_3	(0	0	2	2	7	1	1	1	0	4	0)/2
g_4	(0	0	2	4	10	8	14	6	6	2	0)/2
g_5	(0	0	1	-4	-3	-17	-18	-21	-12	-4	0)/2
g_6	(0	0	1	-7	-8	-27	-24	-28	-15	-8	-2)/2
g_7	(0	1	3	19	34	66	68	58	31	8	2)/2
g_8	(0	0	0	6	18	28	36	24	14	2	0)/2
g_9	(0	0	1	9	8	22	3	5	-8	-4	0)/2
g_{10}	(0	0	1	7	10	30	23	27	10	4	0)/2

Family F20:0b (transpose of F20:0a). Graphs are largest known from $k = 5$. odd-girth maximum from $k = 5$. Maximal levels: $(k + 5)/5$ from $k = 10$.											
LGM odd basis											
$\begin{pmatrix} 2a + 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2a + 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 2a & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2a & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 2a & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 2a & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2a & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 2a & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	2	26	42	93	92	86	46	16	4	0)/2
g_1	(0	1	2	19	19	48	31	30	10	2	0)/2
g_2	(0	0	1	-1	3	7	4	12	2	2	0)/2
g_3	(0	0	2	4	13	17	17	11	6	0	0)/2
g_4	(0	0	2	2	10	6	16	6	4	2	0)/2
g_5	(0	0	1	6	13	17	12	11	-2	0	0)/2
g_6	(0	0	1	9	16	33	36	32	21	8	2)/2
g_7	(0	1	1	17	26	60	56	54	25	8	2)/2
g_8	(0	0	0	6	0	16	4	8	0	-2	0)/2
g_9	(0	0	1	-5	-10	-26	-29	-29	-14	-4	0)/2
g_{10}	(0	0	1	-3	-2	-16	-7	-9	0	0	0)/2

Table A.73: Degree 20, diameter class $k \equiv 1 \pmod{5}$, except $k \equiv 1 \pmod{35}$,
 $a = (k - 1)/5$

Family F20:1a (transpose of F20:1b, conjugate of F20:3a). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 16$. Maximal levels: $(k + 4)/5$ from $k = 11$.

LGM odd basis											
$\begin{pmatrix} 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a+1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a+1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2a & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	6	42	150	337	512	526	352	142	28	0)/2
g_1	(0	1	4	26	80	152	185	145	67	16	0)/2
g_2	(0	0	1	8	13	4	-17	-32	-25	-8	0)/2
g_3	(0	0	1	11	33	72	103	93	47	12	0)/2
g_4	(0	0	2	10	26	38	29	4	-9	-4	0)/2
g_5	(0	0	1	-2	-14	-40	-72	-74	-39	-8	0)/2
g_6	(0	0	1	-2	-9	-6	13	30	27	10	0)/2
g_7	(0	0	1	-2	-24	-77	-143	-173	-136	-64	-14)/2
g_8	(0	1	7	40	126	260	369	353	216	78	14)/2
g_9	(0	0	1	13	62	150	216	193	101	24	0)/2
g_{10}	(0	0	1	8	23	31	27	23	17	6	0)/2

Family F20:1b (transpose of F20:1a, conjugate of F20:3b). Graphs are largest known from $k = 11$. odd-girth maximum from $k = 16$. Maximal levels: $(k + 4)/5$ from $k = 11$.

LGM odd basis											
$\begin{pmatrix} 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 2a+1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 2a+1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 2a & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 2a & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	6	42	150	337	512	526	352	142	28	0)/2
g_1	(0	2	9	70	224	411	478	363	165	38	0)/2
g_2	(0	0	2	8	31	42	8	-28	-25	-6	0)/2
g_3	(0	0	1	4	2	37	98	113	63	18	0)/2
g_4	(0	0	0	2	-29	-120	-204	-196	-107	-26	0)/2
g_5	(0	0	1	0	25	132	285	318	189	50	0)/2
g_6	(0	0	0	8	29	90	180	186	87	12	0)/2
g_7	(0	0	1	13	68	182	305	339	240	94	14)/2
g_8	(0	1	5	29	82	155	207	187	112	48	14)/2
g_9	(0	1	5	31	63	35	-27	-33	-5	2	0)/2
g_{10}	(0	2	14	93	339	722	942	747	333	64	0)/2

Table A.74: Degree 20, diameter class $k \equiv 1 \pmod{5}$, $a = (k-1)/5$

Family F20:1c (transpose of F20:1d, conjugate of F20:3c). Largest known from $k = 36$ for $k \equiv 1 \pmod{35}$. odd-girth maximum from $k = 16$. Maximal levels: $(k+4)/5$ from $k = 6$.

LGM odd basis

$\left(\begin{array}{cccccccccc} 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 2a+1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a+1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a+1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2a+1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 2a & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 2a & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 & -1 & 1 & 2a \end{array} \right)$

Polynomial in $2a$

Order	(1	6	41	144	325	500	535	398	198	60	8)/2
g_1	(0	1	4	26	77	139	168	138	81	30	4)/2
g_2	(0	0	1	8	18	34	50	48	29	10	2)/2
g_3	(0	0	1	10	39	86	109	84	43	14	2)/2
g_4	(0	0	2	10	32	58	79	78	53	24	4)/2
g_5	(0	0	1	-2	-8	-20	-32	-36	-25	-12	-2)/2
g_6	(0	1	6	34	100	197	251	216	121	40	6)/2
g_7	(0	0	1	-1	-15	-53	-103	-120	-85	-32	-4)/2
g_8	(0	0	1	0	0	11	34	41	23	6	0)/2
g_9	(0	0	0	5	25	51	52	24	-5	-10	-2)/2
g_{10}	(0	0	1	8	16	13	-2	-11	-7	-4	-2)/2

Family F20:1d (transpose of F20:1c, conjugate of F20:3d). Largest known from $k = 36$ for $k \equiv 1 \pmod{35}$. odd-girth maximum from $k = 16$. Maximal levels: $(k+4)/5$ from $k = 6$.

LGM odd basis											
$\begin{pmatrix} 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2a+1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2a+1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 2a+1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 & 2a+1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 2a & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 2a & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & -1 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	6	41	144	325	500	535	398	198	60	8)/2
g_1	(0	1	6	34	99	181	212	162	85	28	4)/2
g_2	(0	0	1	0	-10	-38	-64	-62	-41	-16	-2)/2
g_3	(0	0	1	-2	-1	10	25	34	27	12	2)/2
g_4	(0	0	2	6	16	34	55	56	37	18	4)/2
g_5	(0	0	1	10	30	62	76	60	29	10	2)/2
g_6	(0	1	4	26	78	155	207	192	117	42	6)/2
g_7	(0	0	1	11	35	73	109	116	81	30	4)/2
g_8	(0	0	1	10	42	93	128	111	55	12	0)/2
g_9	(0	0	0	5	19	29	20	8	9	8	2)/2
g_{10}	(0	0	1	2	-12	-49	-92	-101	-69	-30	-6)/2

Table A.75: Degree 20, diameter class $k \equiv 2 \pmod{5}$, $a = (k - 2)/5$

Family F20:2a (transpose of F20:2b). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 17$. Maximal levels: $(k + 3)/5$ from $k = 7$.											
LGM odd basis											
$\begin{pmatrix} 2a+2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2a+1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 2a+1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 2a+1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a+1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2a+1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2a+1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	10	70	322	976	1996	2776	2584	1533	518	74)/2
g_1	(0	1	11	75	305	773	1263	1333	877	326	52)/2
g_2	(0	0	0	12	70	196	334	368	260	110	22)/2
g_3	(0	0	3	24	97	240	377	372	219	68	8)/2
g_4	(0	0	3	28	117	290	473	522	379	164	32)/2
g_5	(0	0	3	19	68	170	303	367	278	114	18)/2
g_6	(0	0	1	5	2	-24	-57	-57	-22	6	6)/2
g_7	(0	0	2	6	-18	-146	-394	-576	-484	-220	-42)/2
g_8	(0	0	1	21	131	438	889	1133	885	384	68)/2
g_9	(0	2	17	101	391	992	1673	1857	1293	504	80)/2
g_{10}	(0	1	9	47	165	405	693	811	623	286	60)/2
Family F20:2b (transpose of F20:2a). Graphs are largest known from $k = 12$. odd-girth maximum from $k = 17$. Maximal levels: $(k + 3)/5$ from $k = 7$.											
LGM odd basis											
$\begin{pmatrix} 2a+2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2a+1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 2a+1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 2a+1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2a+1 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 & 2a+1 & 0 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 2a+1 & 0 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	10	70	322	976	1996	2776	2584	1533	518	74)/2
g_1	(0	1	6	37	142	330	478	422	203	34	-5)/2
g_2	(0	0	0	8	80	318	701	936	760	348	69)/2
g_3	(0	0	2	19	79	203	337	360	241	92	15)/2
g_4	(0	0	2	16	48	59	-9	-131	-188	-126	-35)/2
g_5	(0	0	2	22	111	315	551	609	408	145	19)/2
g_6	(0	0	1	12	64	175	286	311	239	123	31)/2
g_7	(0	0	1	19	96	267	472	547	408	183	39)/2
g_8	(0	0	1	-2	-37	-155	-359	-514	-449	-217	-43)/2
g_9	(0	1	10	59	224	560	922	981	640	227	31)/2
g_{10}	(0	1	11	68	271	720	1307	1617	1310	628	135)/2

Table A.76: Degree 20, diameter class $k \equiv 3 \pmod{5}$, except $k \equiv 33 \pmod{35}$,
 $a = (k + 2)/5$

Family F20:3a (transpose of F20:3b, conjugate of F20:1a). Graphs are largest known from $k = 13$. odd-girth maximum from $k = 13$. Maximal levels: $(k + 7)/5$ from $k = 13$.

LGM odd basis											
$\begin{pmatrix} 2a-1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 2a-1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2a-1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a-1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a-1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2a & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	-6	42	-150	337	-512	526	-352	142	-28	0)/2
g_1	(0	2	-9	70	-224	411	-478	363	-165	38	0)/2
g_2	(0	0	2	-8	31	-42	8	28	-25	6	0)/2
g_3	(0	0	1	-4	2	-37	98	-113	63	-18	0)/2
g_4	(0	0	0	2	29	-120	204	-196	107	-26	0)/2
g_5	(0	0	1	0	25	-132	285	-318	189	-50	0)/2
g_6	(0	0	0	8	-29	90	-180	186	-87	12	0)/2
g_7	(0	0	1	-13	68	-182	305	-339	240	-94	14)/2
g_8	(0	1	-5	29	-82	155	-207	187	-112	48	-14)/2
g_9	(0	1	-5	31	-63	35	27	-33	5	2	0)/2
g_{10}	(0	2	-14	93	-339	722	-942	747	-333	64	0)/2

Family F20:3b (transpose of F20:3a, conjugate of F20:1b). Graphs are largest known from $k = 13$. odd-girth maximum from $k = 13$. Maximal levels: $(k + 7)/5$ from $k = 13$.

LGM odd basis											
$\begin{pmatrix} 2a-1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2a-1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2a-1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 2a-1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 2a-1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 2a-1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 2a & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 2a & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	-6	42	-150	337	-512	526	-352	142	-28	0)/2
g_1	(0	1	-4	26	-80	152	-185	145	-67	16	0)/2
g_2	(0	0	1	-8	13	-4	-17	32	-25	8	0)/2
g_3	(0	0	1	-11	33	-72	103	-93	47	-12	0)/2
g_4	(0	0	2	-10	26	-38	29	-4	-9	4	0)/2
g_5	(0	0	1	2	-14	40	-72	74	-39	8	0)/2
g_6	(0	0	1	2	-9	6	13	-30	27	-10	0)/2
g_7	(0	0	1	2	-24	77	-143	173	-136	64	-14)/2
g_8	(0	1	-7	40	-126	260	-369	353	-216	78	-14)/2
g_9	(0	0	1	-13	62	-150	216	-193	101	-24	0)/2
g_{10}	(0	0	1	-8	23	-31	27	-23	17	-6	0)/2

Table A.77: Degree 20, diameter class $k \equiv 3 \pmod{5}$, $a = (k+2)/5$

Family F20:3c (transpose of F20:3d, conjugate of F20:1c). Largest known from $k = 33$ for $k \equiv 33 \pmod{35}$. odd-girth maximum from $k = 13$. Maximal levels: $(k+7)/5$ from $k = 13$.											
LGM odd basis											
	$2a-1$	0	0	-1	-1	-1	-1	-1	0	0	
	0	$2a-1$	0	0	-1	-1	-1	-1	-1	0	
	0	0	$2a-1$	-1	-1	-1	-1	0	0	-1	
	1	0	1	$2a-1$	-1	-1	-1	-1	-1	0	
	1	1	1	1	$2a-1$	0	0	0	0	1	
	1	1	1	1	0	$2a-1$	0	0	0	1	
	1	1	1	1	0	0	$2a$	0	-1	1	
	1	1	0	1	0	0	0	$2a$	0	1	
	0	1	0	1	0	0	1	0	$2a$	1	
	0	0	1	0	-1	-1	-1	-1	1	$2a$	
Polynomial in $2a$											
Order	(1	-6	41	-144	325	-500	535	-398	198	-60	8)/2
g_1	(0	1	-6	34	-99	181	-212	162	-85	28	-4)/2
g_2	(0	0	1	0	-10	38	-64	62	-41	16	-2)/2
g_3	(0	0	1	2	-1	-10	25	-34	27	-12	2)/2
g_4	(0	0	2	-6	16	-34	55	-56	37	-18	4)/2
g_5	(0	0	1	-10	30	-62	76	-60	29	-10	2)/2
g_6	(0	1	-4	26	-78	155	-207	192	-117	42	-6)/2
g_7	(0	0	1	-11	35	-73	109	-116	81	-30	4)/2
g_8	(0	0	1	-10	42	-93	128	-111	55	-12	0)/2
g_9	(0	0	0	5	-19	29	-20	8	-9	8	-2)/2
g_{10}	(0	0	1	-2	-12	49	-92	101	-69	30	-6)/2
Family F20:3d (transpose of F20:3c, conjugate of F20:1d). Largest known from $k = 33$ for $k \equiv 33 \pmod{35}$. odd-girth maximum from $k = 13$. Maximal levels: $(k+7)/5$ from $k = 13$.											
LGM odd basis											
	$2a-1$	0	0	1	1	1	1	1	0	0	
	0	$2a-1$	0	0	1	1	1	1	1	0	
	0	0	$2a-1$	1	1	1	1	0	0	1	
	-1	0	-1	$2a-1$	1	1	1	1	1	0	
	-1	-1	-1	-1	$2a-1$	0	0	0	0	-1	
	-1	-1	-1	-1	0	$2a-1$	0	0	0	-1	
	-1	-1	-1	-1	0	0	$2a$	0	1	-1	
	-1	-1	0	-1	0	0	0	$2a$	0	-1	
	0	-1	0	-1	0	0	-1	0	$2a$	-1	
	0	0	-1	0	1	1	1	1	-1	$2a$	
Polynomial in $2a$											
Order	(1	-6	41	-144	325	-500	535	-398	198	-60	8)/2
g_1	(0	1	-4	26	-77	139	-168	138	-81	30	-4)/2
g_2	(0	0	1	-8	18	-34	50	-48	29	-10	2)/2
g_3	(0	0	1	-10	39	-86	109	-84	43	-14	2)/2
g_4	(0	0	2	-10	32	-58	79	-78	53	-24	4)/2
g_5	(0	0	1	2	-8	20	-32	36	-25	12	-2)/2
g_6	(0	1	-6	34	-100	197	-251	216	-121	40	-6)/2
g_7	(0	0	1	1	-15	53	-103	120	-85	32	-4)/2
g_8	(0	0	1	0	0	-11	34	-41	23	-6	0)/2
g_9	(0	0	0	5	-25	51	-52	24	5	-10	2)/2
g_{10}	(0	0	1	-8	16	-13	-2	11	-7	4	-2)/2

Table A.78: Degree 20, diameter class $k \equiv 4 \pmod{5}$, $a = (k+1)/5$

Family F20:4a (transpose of F20:4b). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 14$. Maximal levels: $(k + 6)/5$ from $k = 9$.											
LGM odd basis											
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 2a & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 2a & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 2a & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2a & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	-2	26	-40	89	-92	77	-44	18	-4	0)/2
g_1	(0	2	-1	51	-30	93	-51	37	-12	2	0)/2
g_2	(0	0	1	4	-7	36	-37	44	-18	12	0)/2
g_3	(0	0	1	-3	50	-82	109	-75	36	-10	0)/2
g_4	(0	0	2	-7	14	-44	48	-53	26	-10	0)/2
g_5	(0	0	1	5	23	13	-9	10	-6	4	0)/2
g_6	(0	0	1	-9	22	-37	39	-25	13	-4	2)/2
g_7	(0	1	-1	17	-18	52	-53	52	-31	14	-2)/2
g_8	(0	0	2	-7	48	-31	38	-26	16	-8	0)/2
g_9	(0	1	-3	25	-36	94	-50	34	-8	4	0)/2
g_{10}	(0	2	-5	48	-74	112	-124	82	-36	12	0)/2
Family F20:4b (transpose of F20:4a). Graphs are largest known from $k = 9$. odd-girth maximum from $k = 19$. Maximal levels: $(k + 6)/5$ from $k = 9$.											
LGM odd basis											
$\begin{pmatrix} 2a-1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2a-1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 2a & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2a & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & -1 & 2a & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 2a & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2a & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2a & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 2a & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 2a \end{pmatrix}$											
Polynomial in $2a$											
Order	(1	-2	26	-40	89	-92	77	-44	18	-4	0)/2
g_1	(0	0	1	-7	12	-31	23	-21	10	-4	0)/2
g_2	(0	0	1	-6	5	-16	9	-8	2	-2	0)/2
g_3	(0	0	1	-3	4	2	-11	7	-2	0	0)/2
g_4	(0	0	2	-3	8	-6	4	9	-8	4	0)/2
g_5	(0	0	1	3	3	1	-3	0	2	-2	0)/2
g_6	(0	0	1	5	-6	19	-19	15	-11	6	-2)/2
g_7	(0	1	-3	21	-34	70	-73	62	-33	12	-2)/2
g_8	(0	0	0	5	-18	31	-38	28	-12	2	0)/2
g_9	(0	1	-1	19	-24	50	-36	22	-6	2	0)/2
g_{10}	(0	0	1	-6	14	-18	26	-20	10	-2	0)/2

APPENDIX B

EXTREMAL AND LARGEST-KNOWN BIPARTITE CIRCULANT GRAPH FAMILIES

This appendix documents extremal and largest-known bipartite graph families up to degree 11. As given by Definition 1.4, a graph family is an infinite set of graphs of given degree d and dimension $f = \lfloor d/2 \rfloor$, defined for each diameter k of a diameter class, with order and generating set specified by polynomials in k of maximum degree f . Unless otherwise stated, the diameter class is modulo f for odd dimension and modulo $f/2$ for even. The graph families are identified by a code, such as D11:4a. In this example, D indicates that it is a bipartite circulant graph family, 11 is the degree, 4 is the diameter class (mod 5), and a is the isomorphism class (where there is more than one).

Of the largest-known bipartite circulant graph families presented in this appendix, the following have been discovered by the author:

Degree 6 and above - all families

All extremal and largest-known bipartite circulant graph families are subquasimaximal with quasimaximal defect 2. Below some low diameter threshold the graphs may not be bipartite or extremal. The extremal graphs are included, up to diameter 16, in Appendix E, with reference to their families by isomorphism class.

B.1 Bipartite circulant graph families of degrees 4 and 5

Table B.1: Degree 4, first two and parametrised families, for diameters k where $\gcd(s, k) = 1$, $a = k$.

Family	LGM odd basis	LGM	Polynomial in $2a$			
D4: s						
D4:1	$\begin{pmatrix} 2a+1 & -1 \\ 1 & 2a-1 \end{pmatrix}$	$\begin{pmatrix} a & -a \\ a+1 & a-1 \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$	$/2$	
			g_1	$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$		
			g_2	$\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$		
D4:2	$\begin{pmatrix} 2a+2 & -2 \\ 2 & 2a-2 \end{pmatrix}$	$\begin{pmatrix} a & -a \\ a+2 & a-2 \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$	$/2$	
			g_1	$\begin{pmatrix} 0 & 1 & -4 \end{pmatrix}$	$/2$	
			g_2	$\begin{pmatrix} 0 & 1 & 4 \end{pmatrix}$	$/2$	
D4: s s odd	$\begin{pmatrix} 2a+s & -s \\ s & 2a-s \end{pmatrix}$	$\begin{pmatrix} a & -a \\ a+s & a-s \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$	$/2$	
			g_1	$\begin{pmatrix} 0 & 0 & s \end{pmatrix}$		
			g_2	$\begin{pmatrix} 0 & 1 & -s \end{pmatrix}$		
D4: s s even	$\begin{pmatrix} 2a+s & -s \\ s & 2a-s \end{pmatrix}$	$\begin{pmatrix} a & -a \\ a+s & a-s \end{pmatrix}$	Order	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$	$/2$	
			g_1	$\begin{pmatrix} 0 & 1 & -2s \end{pmatrix}$		
			g_2	$\begin{pmatrix} 0 & 1 & 2s \end{pmatrix}$		

All D4: s families are self-transpose and self-conjugate

Table B.2: Degree 5, all diameters, $a = k$

LGM	Polynomial in $2a$
Family D5 (self-transpose, self-conjugate)	
$\begin{pmatrix} 2a-1 & -1 \\ 1 & 2a-1 \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
	g_1
	g_2

B.2 Bipartite circulant graph families of degrees 6 and 7**Table B.3:** Degree 6, diameter class 0 (mod 3), $a = 2k/3$

LGM odd basis	LGM	Polynomial in $2a$
Family D6:0 (self-transpose, self-conjugate, translate of D7:2)		
$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	$\begin{pmatrix} a-1 & -a-1 & -a \\ a & a-1 & -a-1 \\ a & -a & a \end{pmatrix}$	Order $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} /2$
		g_1
		g_2
		g_3

Table B.4: Degree 6, diameter class 1 (mod 3), $a = (2k+1)/3$

LGM odd basis	LGM	Polynomial in $2a$
Family D6:1A (self-transpose, conjugate of D6:2A, translate of D7:0A)		
$\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	$\begin{pmatrix} a-1 & -a-1 & -a+1 \\ a & a-1 & -a \\ a-1 & -a & a \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} /2$
		g_1
		g_2
		g_3
Family D6:1B (self-transpose, conjugate of D6:2B, translate of D7:0B)		
$\begin{pmatrix} 2a-2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	$\begin{pmatrix} a-2 & -a-1 & -a \\ a-1 & a-1 & -a-1 \\ a-1 & -a & a \end{pmatrix}$	Order $\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 4 \\ 0 & 1 & -1 & 0 \end{pmatrix} /2$
		g_1
		g_2
		g_3

Table B.5: Degree 6, diameter class 2 (mod 3), $a = (2k-1)/3$

LGM odd basis	LGM	Polynomial in $2a$
Family D6:2A (self-transpose, conjugate of D6:1A, translate of D7:1A)		
$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$	$\begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a & -a & a+1 \end{pmatrix}$	Order $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} /2$
		g_1
		g_2
		g_3
Family D6:2B (self-transpose, conjugate of D6:1B, translate of D7:1B)		
$\begin{pmatrix} 2a+2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	$\begin{pmatrix} a & -a-1 & -a \\ a+1 & a-1 & -a-1 \\ a+1 & -a & a \end{pmatrix}$	Order $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix} /2$
		g_1
		g_2
		g_3

Table B.6: Degree 7, diameter class 0 (mod 3), $a = 2k/3$

LGM	Polynomial in $2a$			
Family D7:0A (self-transpose, conjugate of D7:1A, translate of D6:1A)				
$\begin{pmatrix} 2a-1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a-1 \end{pmatrix}$	Order	(1	-2	3 -2)
	g_1	(0	0	0 1)
	g_2	(0	0	1 -1)
	g_3	(0	1	-1 1)
Family D7:0B (self-transpose, conjugate of D7:1B, translate of D6:1B)				
$\begin{pmatrix} 2a-2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	Order	(1	-2	3 -2)
	g_1	(0	0	0 1)
	g_2	(1	-3	2 -4) /4
	g_3	(1	-3	6 -4) /4

Table B.7: Degree 7, diameter class 1 (mod 3), $a = (2k-2)/3$

LGM	Polynomial in $2a$				
Family D7:1A (self-transpose, conjugate of D7:0A, translate of D6:2A)					
$\begin{pmatrix} 2a+1 & -1 & 0 \\ 1 & 2a & -1 \\ 0 & 1 & 2a+1 \end{pmatrix}$	Order	(1	2	3	2)
	g_1	(0	0	0	1)
	g_2	(0	0	1	1)
	g_3	(0	1	1	1)
Family D7:1B (self-transpose, conjugate of D7:0B, translate of D6:2B)					
$\begin{pmatrix} 2a+2 & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	Order	(1	2	3	2)
	g_1	(0	0	0	1)
	g_2	(1	3	2	4) /4
	g_3	(1	3	6	4) /4

Table B.8: Degree 7, diameter class 2 (mod 3), $a = (2k-1)/3$

LGM	Polynomial in $2a$			
Family D7:2 (self-transpose, self-conjugate, translate of D6:0)				
$\begin{pmatrix} 2a & -1 & -1 \\ 1 & 2a & -1 \\ 1 & 1 & 2a \end{pmatrix}$	Order	(1	0	3 0)
	g_1	(0	0	0 1)
	g_2	(1	-1	2 -2) /2
	g_3	(1	-1	4 -2) /2

B.3 Bipartite circulant graph families of degrees 8 and 9

Table B.9: Degree 8, diameter class 0 (mod 2), $a = k/2$

LGM odd basis	Polynomial in $2a$						
Family D8:0 (self-transpose)							
$\begin{pmatrix} 2a+1 & 0 & 0 & -1 \\ 0 & 2a & -1 & -1 \\ 0 & 1 & 2a & -1 \\ 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order	(1	0	3	2	0)	/2
	g_1	(0	0	0	0	1)	
	g_2	(0	0	0	1	1)	
	g_3	(0	1	-1	2	2)	/2
	g_4	(0	1	1	2	2)	/2

Table B.10: Degree 8, diameter class 1 (mod 2), $a = (k+1)/2$

LGM odd basis	Polynomial in $2a$						
Family D8:1 (self-transpose)							
$\begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a-1 & -1 & -1 \\ 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 2a-2 \end{pmatrix}$	Order	(1	-4	10	-12	4)	/2
	g_1	(0	0	0	0	1)	
	g_2	(0	1	-4	10	-10)	/2
	g_3	(0	1	-3	8	-6)	/2
	g_4	(0	2	-7	16	-14)	/2

Table B.11: Degree 9, diameter class 0 (mod 2), $a = k/2$

LGM	Polynomial in $2a$						
Family D9:0 (self-transpose, conjugate of D9:1)							
$\left(\begin{array}{cccc} 2a-1 & 0 & 0 & -1 \\ 0 & 2a & -1 & -1 \\ 0 & 1 & 2a & -1 \\ 1 & 1 & 1 & 2a-1 \end{array}\right)$	Order	(1	-2	5	-4	2)	
	g_1	(0	0	0	0	1)	
	g_2	(0	0	0	1	-1)	
	g_3	(1	-3	6	-6	2)	/2
	g_4	(1	-3	8	-10	6)	/2

Table B.12: Degree 9, diameter class 1 (mod 2), $a = (k+1)/2$

LGM	Polynomial in $2a$						
Family D9:1 (self-transpose, conjugate of D9:0)							
$\begin{pmatrix} 2a-1 & 0 & 0 & -1 \\ 0 & 2a-2 & -1 & -1 \\ 0 & 1 & 2a-2 & -1 \\ 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order	(1	-6	17	-24	14)	
	g_1	(0	0	0	0	1)	
	g_2	(0	0	0	1	-1)	
	g_3	(1	-7	20	-30	18)	/2
	g_4	(1	-7	22	-34	22)	/2

B.4 Bipartite circulant graph families of degrees 10 and 11**Table B.13:** Degree 10, diameter class 0 (mod 5), $a = 2k/5$

LGM odd basis	Polynomial in $2a$							
Family D10:0 (self-transpose, self-conjugate, translate of D11:3)								
$\left(\begin{array}{ccccc} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a & 0 & 0 \\ 1 & 0 & 0 & 2a & -1 \\ 1 & 0 & 0 & 1 & 2a \end{array}\right)$	Order	(1	0	6	0	5	0)	/2
	g_1	(0	0	0	2	0	2)	/2
	g_2	(0	1	0	5	2	2)	/2
	g_3	(0	0	1	0	3	2)	/2
	g_4	(0	0	1	0	3	-2)	/2
	g_5	(0	1	0	5	-2	2)	/2

Table B.14: Degree 10, diameter class 1 (mod 5), $a = (2k - 2)/5$

LGM odd basis					Polynomial in $2a$						
Family D10:1 (self-transpose, conjugate of D10:4)											
$\left(\begin{array}{ccccc} 2a & -1 & -1 & 0 & 0 \\ 1 & 2a+1 & 0 & -1 & -1 \\ 1 & 0 & 2a+1 & -1 & -1 \\ 0 & 1 & 1 & 2a+1 & -1 \\ 0 & 1 & 1 & 1 & 2a+1 \end{array}\right)$	Order	(1	4	13	20	14	4)	/2			
	g_1	(0	2	11	33	46	22)	/2			
	g_2	(0	1	2	-2	-8	-6)	/2			
	g_3	(0	2	5	8	2	-2)	/2			
	g_4	(0	0	2	5	-7	-10)	/2			
	g_5	(0	1	0	8	11	2)	/2			

Table B.15: Degree 10, diameter class 2 (mod 5), $a = (2k + 1)/5$

LGM odd basis	Polynomial in $2a$							
Family D10:2 (self-transpose, conjugate of D10:3, translate of D11:0)								
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & 0 \\ 1 & 2a-1 & 1 & 0 & -1 \\ 1 & -1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order	(1	-2	8	-8	5	-2)	/2
	g_1	(0	1	0	-2	2	-1)	
	g_2	(0	1	-4	4	-2	1)	
	g_3	(0	1	2	-2	2	-1)	
	g_4	(0	1	-4	6	-4	1)	
	g_5	(0	1	-2	2	-2	1)	

Table B.16: Degree 10, diameter class 3 (mod 5), $a = (2k - 1)/5$

LGM odd basis	Polynomial in $2a$							
Family D10:3 (self-transpose, conjugate of D10:2, translate of D11:1)								
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & 1 & 0 & -1 \\ 1 & -1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order	(1	2	8	8	5	2)	/2
	g_1	(0	1	4	8	6	3)	
	g_2	(0	1	2	2	4	1)	
	g_3	(0	1	0	0	0	1)	
	g_4	(0	1	2	4	0	-1)	
	g_5	(0	1	0	6	6	1)	

Table B.17: Degree 10, diameter class 4 (mod 5), $a = (2k + 2)/5$

LGM odd basis					Polynomial in $2a$						
Family D10:4 (self-transpose, conjugate of D10:1)											
$\begin{pmatrix} 2a & -1 & -1 & 0 & 0 \\ 1 & 2a-1 & 0 & -1 & -1 \\ 1 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order	(1	-4	13	-20	14	-4)	/2			
	g_1	(0	2	-7	25	-38	18)	/2			
	g_2	(0	0	1	-4	0	2)	/2			
	g_3	(0	1	-2	6	-10	6)	/2			
	g_4	(0	2	-6	15	-13	2)	/2			
	g_5	(0	1	-6	14	-23	14)	/2			

Table B.18: Degree 11, diameter class 0 (mod 5), $a = 2k/5$

LGM					Polynomial in $2a$					
Family D11:0 (self-transpose, conjugate of D11:1, translate of D10:2)										
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & 0 \\ 1 & 2a-1 & 1 & 0 & -1 \\ 1 & -1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{pmatrix}$	Order	(1	-2	8	-8	5	-2)			
	g_1	(0	1	0	-2	2	-1)			
	g_2	(0	1	-4	4	-2	1)			
	g_3	(0	1	2	-2	2	-1)			
	g_4	(0	1	-4	6	-4	1)			
	g_5	(0	1	-2	2	-2	1)			

Table B.19: Degree 11, diameter class 1 (mod 5), $a = (2k - 2)/5$

LGM					Polynomial in $2a$						
Family D11:1 (self-transpose, conjugate of D11:0, translate of D10:3)											
$\left(\begin{array}{ccccc} 2a+1 & -1 & -1 & -1 & 0 \\ 1 & 2a+1 & 1 & 0 & -1 \\ 1 & -1 & 2a & 1 & -1 \\ 1 & 0 & -1 & 2a & 0 \\ 0 & 1 & 1 & 0 & 2a \end{array}\right)$	Order	(1	2	8	8	5	2)				
	g_1	(0	1	4	8	6	3)				
	g_2	(0	1	2	2	4	1)				
	g_3	(0	1	0	0	0	1)				
	g_4	(0	1	2	4	0	-1)				
	g_5	(0	1	0	6	6	1)				

Table B.20: Degree 11, diameter class 2 (mod 5), $a = (2k + 1)/5$

LGM					Polynomial in $2a$						
Family D11:2 (self-transpose)											
$\begin{pmatrix} 2a+1 & 0 & -1 & 0 & -1 \\ 0 & 2a+1 & 0 & -1 & -1 \\ 1 & 0 & 2a & -1 & -1 \\ 0 & 1 & 1 & 2a & 1 \\ 1 & 1 & 1 & -1 & 2a-1 \end{pmatrix}$	Order	(1	-4	12	-16	9	-4)				
	g_1	(0	1	-4	8	-12	5)				
	g_2	(0	1	-4	8	-4	-3)				
	g_3	(0	1	-4	8	-4	5)				
	g_4	(0	1	-4	16	-12	5)				
	g_5	(0	1	0	4	0	1)				

Table B.21: Degree 11, diameter class 3 (mod 5), $a = (2k - 1)/5$

LGM	Polynomial in $2a$						
Family D11:3 (self-transpose, self-conjugate, translate of D10:0)							
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a & 0 & 0 \\ 1 & 0 & 0 & 2a & -1 \\ 1 & 0 & 0 & 1 & 2a \end{pmatrix}$	Order	(1	0	6	0	5	0)
	g_1	(0	1	0	2	0	1)
	g_2	(0	1	0	4	4	-1)
	g_3	(0	1	-2	6	-6	1)
	g_4	(0	1	2	6	6	1)
	g_5	(0	1	0	4	-4	-1)

Table B.22: Degree 11, diameter class 4 (mod 5), $a = (2k - 3)/5$

LGM	Polynomial in $2a$						
Family D11:4a (transpose of D11:4b)							
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & -1 & 0 & 0 \\ 1 & 1 & 2a+1 & 1 & 0 \\ 1 & 0 & -1 & 2a+1 & -1 \\ 1 & 0 & 0 & 1 & 2a+2 \end{pmatrix}$	Order	(1	4	12	20	15	4)
	g_1	(0	1	4	12	16	7)
	g_2	(0	1	4	10	14	3)
	g_3	(0	1	2	6	4	3)
	g_4	(0	1	4	12	22	9)
	g_5	(0	1	2	8	4	1)
Family D11:4b (transpose of D11:4a)							
$\begin{pmatrix} 2a & 1 & 1 & 1 & 1 \\ -1 & 2a & 1 & 0 & 0 \\ -1 & -1 & 2a+1 & -1 & 0 \\ -1 & 0 & 1 & 2a+1 & 1 \\ -1 & 0 & 0 & -1 & 2a+2 \end{pmatrix}$	Order	(1	4	12	20	15	4)
	g_1	(0	1	6	18	26	13)
	g_2	(0	1	4	8	12	5)
	g_3	(0	1	2	10	16	9)
	g_4	(0	1	2	4	4	3)
	g_5	(0	1	2	4	-2	-3)

APPENDIX C

LARGEST-KNOWN ABELIAN CAYLEY GRAPH FAMILIES (IF NON-CIRCULANT)

This appendix documents extremal and largest-known Abelian Cayley graphs that are not circulant, up to degree 19. As given by Definition 1.4, a graph family is an infinite set of graphs of given degree d and dimension $f = \lfloor d/2 \rfloor$, defined for each diameter k of a diameter class, with order and generating set specified by polynomials in k of maximum degree f . Unless otherwise stated, the diameter class is modulo f for odd dimension and modulo $f/2$ for even. The graph families are identified by a code, such as A14:5a. In this example, A indicates that it is an Abelian Cayley graph family that is not circulant, 14 is the degree, 5 is the diameter class (mod 7), and a is the isomorphism class (where there is more than one).

Of the largest-known Abelian Cayley graph families presented in this appendix, the following have been discovered by the author:

Degree 6 and above - all families

All extremal and largest-known non-circulant Abelian Cayley graph families are quasimaximal, with maximum odd girth above some low diameter threshold (often zero). The extremal and largest-known graphs are included, up to diameter 16, in Appendix F, with reference to their families by isomorphism class.

C.1 Abelian Cayley graph families of degree 5

Table C.1: Degree 5, all diameters k , $a = k$

LGM	Polynomial in $2a$		
Family A5 (self-transpose, self-conjugate)			
Cyclic rank 2 (suffices a and b)			
$\begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix}$	Order, n	$(1 \quad 0 \quad 0)$	
	n_a	$(0 \quad 1 \quad 0)$	
	n_b	$(0 \quad 1 \quad 0)$	
	g_{1a}	$(0 \quad 0 \quad 0)$	
	g_{1b}	$(0 \quad 0 \quad 1)$	
	g_{2a}	$(0 \quad 0 \quad 1)$	
	g_{2b}	$(0 \quad 0 \quad 0)$	
	g_{ma}	$(0 \quad 1 \quad 0)$	$/2$
	g_{mb}	$(0 \quad 1 \quad 0)$	$/2$

C.2 Abelian Cayley graph families of degree 9

Table C.2: Degree 9, diameter class 0 (mod 2), $a = k/2$

LGM	Polynomial in $2a$						
Family A9:0Aa0, for $k \equiv 0 \pmod{4}$ (transpose of A9:0Ab0, self-conjugate)							
Cyclic rank 2 (suffices a and b)							
$\begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & -1 \\ 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 2a \end{pmatrix}$	Order, n	(1	0	4	0	0)	
	n_a	(0	1	0	4	0)	/2
	n_b	(0	0	0	2	0)	
	g_{1a}	(0	1	-2	4	4)	/4
	g_{1b}	(0	0	0	1	-1)	
	g_{2a}	(0	1	0	-2	0)	/4
	g_{2b}	(0	0	0	1	-1)	
	g_{3a}	(0	1	0	-2	0)	/4
	g_{3b}	(0	0	0	0	1)	
	g_{4a}	(0	1	-4	4	-4)	/4
	g_{4b}	(0	0	0	1	-1)	
	g_{ma}	(0	1	0	4	0)	/4
	g_{mb}	(0	0	0	1	0)	
Family A9:0Aa2, for $k \equiv 2 \pmod{4}$ (transpose of A9:0Ab2, self-conjugate)							
Cyclic rank 2 (suffices a and b)							
$\begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a+1 & -1 & -1 \\ 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 2a \end{pmatrix}$	Order, n	(1	0	4	0	0)	
	n_a	(0	1	0	4	0)	/2
	n_b	(0	0	0	2	0)	
	g_{1a}	(0	1	-2	4	4)	/4
	g_{1b}	(0	0	0	1	-1)	
	g_{2a}	(0	1	0	-2	0)	/4
	g_{2b}	(0	0	0	1	-2)	
	g_{3a}	(0	1	0	-2	0)	/4
	g_{3b}	(0	0	0	0	2)	
	g_{4a}	(0	1	4	4	4)	/4
	g_{4b}	(0	0	0	0	3)	
	g_{ma}	(0	1	0	4	0)	/4
	g_{mb}	(0	0	0	1	0)	
Family A9:0Ab0, for $k \equiv 0 \pmod{4}$ (transpose of A9:0Aa0, self-conjugate)							
Cyclic rank 2 (suffices a and b)							
$\begin{pmatrix} 2a & 1 & 1 & 0 \\ -1 & 2a+1 & 1 & 1 \\ -1 & -1 & 2a-1 & 1 \\ 0 & -1 & -1 & 2a \end{pmatrix}$	Order, n	(1	0	4	0	0)	
	n_a	(0	1	0	4	0)	/2
	n_b	(0	0	0	2	0)	
	g_{1a}	(0	3	2	8	0)	/8
	g_{1b}	(0	0	0	3	2)	/2
	g_{2a}	(0	3	2	12	-8)	/8
	g_{2b}	(0	0	0	3	2)	/2
	g_{3a}	(0	3	2	20	8)	/8
	g_{3b}	(0	0	0	3	-2)	/2
	g_{4a}	(0	3	-2	16	0)	/8
	g_{4b}	(0	0	0	3	2)	/2
	g_{ma}	(0	1	0	4	0)	/4
	g_{mb}	(0	0	0	1	0)	

continued on next page

Table C.2: (cont.) Degree 9, diameter class 0 (mod 2), $a = k/2$

LGM	Polynomial in $2a$						
Family A9:0Ab2, for $k \equiv 2 \pmod{4}$ (transpose of A9:0Aa2, self-conjugate)							
Cyclic rank 2 (suffices a and b)							
$\begin{pmatrix} 2a & 1 & 1 & 0 \\ -1 & 2a+1 & 1 & 1 \\ -1 & -1 & 2a-1 & 1 \\ 0 & -1 & -1 & 2a \end{pmatrix}$	Order, n	(1	0	4	0	0)	
	n_a	(0	1	0	4	0)	/2
	n_b	(0	0	0	2	0)	
	g_{1a}	(0	3	0	16	0)	/8
	g_{1b}	(0	0	0	3	-2)	/2
	g_{2a}	(0	3	0	16	8)	/8
	g_{2b}	(0	0	0	3	0)	/2
	g_{3a}	(0	3	-4	8	-8)	/8
	g_{3b}	(0	0	0	1	0)	/2
	g_{4a}	(0	3	0	8	0)	/8
	g_{4b}	(0	0	0	1	-2)	/2
	g_{ma}	(0	1	0	4	0)	/4
	g_{mb}	(0	0	0	0	0)	
Family A9:0B0, for $k \equiv 0 \pmod{4}$ (self-transpose, self-conjugate)							
Cyclic rank 2 (suffices a and b)							
$\begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & -1 \\ 0 & 1 & 1 & 2a \end{pmatrix}$	Order, n	(1	0	4	0	0)	
	n_a	(0	1	0	4	0)	/2
	n_b	(0	0	0	2	0)	
	g_{1a}	(0	3	-2	12	8)	/8
	g_{1b}	(0	0	0	3	0)	/2
	g_{2a}	(0	3	0	20	0)	/8
	g_{2b}	(0	0	0	3	2)	/2
	g_{3a}	(0	3	0	20	0)	/8
	g_{3b}	(0	0	0	1	-2)	/2
	g_{4a}	(0	3	6	12	8)	/8
	g_{4b}	(0	0	0	1	0)	/2
	g_{ma}	(0	1	0	4	0)	/4
	g_{mb}	(0	0	0	1	0)	
Family A9:0B2, for $k \equiv 2 \pmod{4}$ (self-transpose, self-conjugate)							
Cyclic rank 2 (suffices a and b)							
$\begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a & 0 & -1 \\ 1 & 0 & 2a & -1 \\ 0 & 1 & 1 & 2a \end{pmatrix}$	Order, n	(1	0	4	0	0)	
	n_a	(0	1	0	4	0)	/2
	n_b	(0	0	0	2	0)	
	g_{1a}	(0	3	-4	12	-8)	/8
	g_{1b}	(0	0	0	3	-2)	/2
	g_{2a}	(0	3	-2	16	-8)	/8
	g_{2b}	(0	0	0	3	0)	/2
	g_{3a}	(0	3	2	16	8)	/8
	g_{3b}	(0	0	0	1	0)	/2
	g_{4a}	(0	3	0	12	-8)	/8
	g_{4b}	(0	0	0	1	-2)	/2
	g_{ma}	(0	1	0	4	0)	/4
	g_{mb}	(0	0	0	0	0)	

Table C.3: Degree 9, diameter class 1 (mod 2), $a = (k + 1)/2$

LGM				Polynomial in $2a$				
Family A9:1 (self-transpose, self-conjugate)								
Cyclic rank 2 (suffices a and b)								
$\begin{pmatrix} 2a & -1 & -1 & 0 \\ 1 & 2a-1 & -1 & -1 \\ 1 & 1 & 2a-1 & -1 \\ 0 & 1 & 1 & 2a-2 \end{pmatrix}$	Order, n	(1	4	10	12	4)		
	n_a	(1	4	10	12	4)	/2	
	n_b	(0	0	0	0	2)		
	g_{1a}	(1	2	4	8	4)	/4	
	g_{1b}	(0	0	0	0	1)		
	g_{2a}	(1	2	8	16	8)	/4	
	g_{2b}	(0	0	0	0	1)		
	g_{3a}	(1	4	16	24	8)	/4	
	g_{3b}	(0	0	0	0	1)		
	g_{4a}	(1	6	20	28	12)	/4	
	g_{4b}	(0	0	0	0	1)		
	g_{ma}	(1	4	10	12	4)	/4	
	g_{mb}	(0	0	0	0	0)		

C.3 Abelian Cayley graph families of degrees 10 and 11

Table C.4: Degree 10, diameter class 1 (mod 5), $a = (2k + 3)/5$

LGM odd basis					Polynomial in $2a$							
Family A10:1 (self-transpose, conjugate of A10:3, translate of A11:4)												
Cyclic rank 2 (suffices a and b)												
$\begin{pmatrix} 2a-1 & -1 & -1 & 0 & -1 \\ 1 & 2a-1 & 0 & -1 & -1 \\ 1 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 2a-1 & 1 \\ 1 & 1 & 1 & -1 & 2a \end{pmatrix}$					Order, n	(1	-4	14	-24	17	-4)	/2
					n_a	(0	1	-3	11	-13	4)	/2
					n_b	(0	0	0	0	1	-1)	
					g_{1a}	(0	0	1	-2	6	-3)	
					g_{1b}	(0	0	0	0	0	1)	
					g_{2a}	(0	0	0	1	-2	1)	
					g_{2b}	(0	0	0	0	1	-2)	
					g_{3a}	(0	0	0	1	-2	1)	
					g_{3b}	(0	0	0	0	0	1)	
					g_{4a}	(0	0	0	0	3	-1)	
					g_{4b}	(0	0	0	0	0	1)	
					g_{5a}	(0	0	0	1	0	-1)	
					g_{5b}	(0	0	0	0	0	0)	

Table C.5: Degree 10, diameter class 2 (mod 5), $a = (2k + 1)/5$

LGM odd basis	Polynomial in $2a$							
Family A10:2 (self-transpose, self-conjugate, translate of A11:0)								
Cyclic rank 3 (suffices a , b and c)								
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & 0 & 0 & -1 \\ 1 & 0 & 2a & 0 & -1 \\ 1 & 0 & 0 & 2a & -1 \\ 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order, n	(1	0	7	0	0	0)	/2
	n_a	(0	0	1	0	7	0)	
	n_b	(0	0	0	0	1	0)	
	n_c	(0	0	0	0	1	0)	/2
	g_{1a}	(0	0	1	0	17	10)	/2
	g_{1b}	(0	0	0	0	1	-2)	/2
	g_{1c}	(0	0	0	0	1	-2)	/4
	g_{2a}	(0	0	0	1	5	-8)	/2
	g_{2b}	(0	0	0	0	0	0)	
	g_{2c}	(0	0	0	0	0	0)	
	g_{3a}	(0	0	0	1	5	-8)	/2
	g_{3b}	(0	0	0	0	0	1)	
	g_{3c}	(0	0	0	0	0	0)	
	g_{4a}	(0	0	0	3	5	6)	/2
	g_{4b}	(0	0	0	0	1	0)	/2
	g_{4c}	(0	0	0	0	1	2)	/4
	g_{5a}	(0	0	0	5	-5	10)	/2
	g_{5b}	(0	0	0	0	1	-2)	/2
	g_{5c}	(0	0	0	0	1	-2)	/4

Table C.6: Degree 10, diameter class 3 (mod 5), $a = (2k - 1)/5$

LGM odd basis	Polynomial in $2a$							
Family A10:3 (self-transpose, conjugate of A10:1, translate of A11:1)								
Cyclic rank 2 (suffices a and b)								
$\begin{pmatrix} 2a+1 & -1 & -1 & 0 & -1 \\ 1 & 2a+1 & 0 & -1 & -1 \\ 1 & 0 & 2a+1 & -1 & -1 \\ 0 & 1 & 1 & 2a+1 & 1 \\ 1 & 1 & 1 & -1 & 2a \end{pmatrix}$	Order, n	(1	4	14	24	17	4)	/2
	n_a	(0	1	3	11	13	4)	/2
	n_b	(0	0	0	0	1	1)	
	g_{1a}	(0	0	1	2	6	3)	
	g_{1b}	(0	0	0	0	0	1)	
	g_{2a}	(0	0	0	1	2	1)	
	g_{2b}	(0	0	0	0	1	0)	
	g_{3a}	(0	0	0	1	2	1)	
	g_{3b}	(0	0	0	0	0	1)	
	g_{4a}	(0	0	0	0	3	1)	
	g_{4b}	(0	0	0	0	0	1)	
	g_{5a}	(0	0	0	1	0	-1)	
	g_{5b}	(0	0	0	0	0	0)	

Table C.7: Degree 11, diameter class 0 (mod 5), $a = 2k/5$

LGM	Polynomial in $2a$							
Family A11:0 (self-transpose, self-conjugate, translate of A10:2)								
Cyclic rank 3 (suffices a , b and c)								
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 \\ 1 & 2a & 0 & 0 & -1 \\ 1 & 0 & 2a & 0 & -1 \\ 1 & 0 & 0 & 2a & -1 \\ 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order, n	(1	0	7	0	0	0)	
	n_a	(0	0	1	0	7	0)	
	n_b	(0	0	0	0	1	0)	
	n_c	(0	0	0	0	1	0)	
	g_{1a}	(0	0	0	1	4	3)	
	g_{1b}	(0	0	0	0	1	-1)	
	g_{1c}	(0	0	0	0	1	-1)	
	g_{2a}	(0	0	0	1	2	-3)	
	g_{2b}	(0	0	0	0	0	1)	
	g_{2c}	(0	0	0	0	0	1)	
	g_{3a}	(0	0	0	1	-2	1)	
	g_{3b}	(0	0	0	0	1	-1)	
	g_{3c}	(0	0	0	0	0	0)	
	g_{4a}	(0	0	0	1	-2	1)	
	g_{4b}	(0	0	0	0	0	0)	
	g_{4c}	(0	0	0	0	1	-1)	
	g_{5a}	(0	0	0	1	-2	1)	
	g_{5b}	(0	0	0	0	0	0)	
	g_{5c}	(0	0	0	0	0	0)	
	g_{ma}	(0	0	1	0	7	0)	/2
	g_{mb}	(0	0	0	0	1	0)	/2
	g_{mc}	(0	0	0	0	1	0)	/2

Table C.8: Degree 11, diameter class 1 (mod 5), $a = (2k - 2)/5$

LGM	Polynomial in $2a$						
Family A11:1 (self-transpose, conjugate of A11:4, translate of A10:3)							
Cyclic rank 2 (suffices a and b)							
$\left(\begin{array}{ccccc} 2a+1 & -1 & -1 & 0 & -1 \\ 1 & 2a+1 & 0 & -1 & -1 \\ 1 & 0 & 2a+1 & -1 & -1 \\ 0 & 1 & 1 & 2a+1 & 1 \\ 1 & 1 & 1 & -1 & 2a \end{array}\right)$	Order, n	(1	4	14	24	17	4)
	n_a	(0	1	3	11	13	4)
	n_b	(0	0	0	0	1	1)
	g_{1a}	(0	0	1	5	13	5)
	g_{1b}	(0	0	0	0	1	0)
	g_{2a}	(0	0	1	3	3	1)
	g_{2b}	(0	0	0	0	1	0)
	g_{3a}	(0	0	1	3	3	1)
	g_{3b}	(0	0	0	0	0	1)
	g_{4a}	(0	0	1	-1	5	3)
	g_{4b}	(0	0	0	0	1	0)
	g_{5a}	(0	0	1	1	-1	-1)
	g_{5b}	(0	0	0	0	0	0)
	g_{ma}	(0	1	3	11	13	4)
	g_{mb}	(0	0	0	0	0	0)
		/2					

Table C.9: Degree 11, diameter class 4 (mod 5), $a = (2k + 2)/5$

LGM	Polynomial in $2a$					
Family A11:4 (self-transpose, conjugate of A11:1, translate of A10:1)						
Cyclic rank 2 (suffices a and b)						
$\begin{pmatrix} 2a-1 & -1 & -1 & 0 & -1 \\ 1 & 2a-1 & 0 & -1 & -1 \\ 1 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 2a-1 & 1 \\ 1 & 1 & 1 & -1 & 2a \end{pmatrix}$	Order, n	(1	-4	14	-24	17 -4)
	n_a	(0	1	-3	11	-13 4)
	n_b	(0	0	0	0	1 -1)
	g_{1a}	(0	0	1	1	5 -3)
	g_{1b}	(0	0	0	0	0 1)
	g_{2a}	(0	0	1	-1	-1 1)
	g_{2b}	(0	0	0	0	0 1)
	g_{3a}	(0	0	1	-1	-1 1)
	g_{3b}	(0	0	0	0	1 -2)
	g_{4a}	(0	0	1	-5	13 -5)
	g_{4b}	(0	0	0	0	0 1)
	g_{5a}	(0	0	1	-5	7 -3)
	g_{5b}	(0	0	0	0	0 0)
	g_{ma}	(0	1	-3	11	-13 4) /2
	g_{mb}	(0	0	0	0	0 0)

C.4 Abelian Cayley graph families of degrees 12 and 13

Table C.10: Degree 12, diameter class 0 (mod 3), $a = k/3$

LGM odd basis	Polynomial in $2a$							
Family A12:0 (self-transpose, conjugate of A12:2)								
Cyclic rank 2 (suffices a and b)								
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a & 0 & -1 & -1 \\ 1 & 1 & 0 & 2a & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 \\ 0 & 1 & 1 & 1 & 0 & 2a+1 \end{pmatrix}$	Order, n	(1	2	12	16	10	0	0) /2
	n_a	(0	1	2	12	16	10	0) /2
	n_b	(0	0	0	0	0	1	0)
	g_{1a}	(0	0	0	1	2	-5	0) /2
	g_{1b}	(0	0	0	0	0	0	0)
	g_{2a}	(0	0	0	2	3	3	-2) /2
	g_{2b}	(0	0	0	0	0	0	0)
	g_{3a}	(0	0	0	1	-4	-6	-4) /2
	g_{3b}	(0	0	0	0	0	1	-1)
	g_{4a}	(0	0	1	3	8	10	6) /2
	g_{4b}	(0	0	0	0	0	0	1)
	g_{5a}	(0	0	0	1	7	9	2) /2
	g_{5b}	(0	0	0	0	0	0	0)
	g_{6a}	(0	0	1	0	6	3	0) /2
	g_{6b}	(0	0	0	0	0	0	0)

Table C.11: Degree 12, diameter class 1 (mod 3), $a = (k - 1)/3$

LGM odd basis	Polynomial in $2a$
Family A12:1a (transpose of A12:1b, self-conjugate)	
Cyclic rank 2 (suffices a and b)	
$\begin{pmatrix} 2a+2 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2a+1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2a+1 \end{pmatrix}$	Order, n (1 6 25 60 72 40 8) /2 n_a (0 1 5 20 40 32 8) /2 n_b (0 0 0 0 0 1 1) g_{1a} (0 0 1 6 11 10 4) /2 g_{1b} (0 0 0 0 0 0 0) g_{2a} (0 0 0 3 3 -8 -8) /2 g_{2b} (0 0 0 0 0 0 0) g_{3a} (0 0 1 4 9 11 5) g_{3b} (0 0 0 0 0 0 0) g_{4a} (0 0 0 3 14 18 6) /2 g_{4b} (0 0 0 0 0 0 1) g_{5a} (0 0 0 3 14 18 6) /2 g_{5b} (0 0 0 0 0 0 0) g_{6a} (0 0 1 1 2 6 2) /2 g_{6b} (0 0 0 0 0 0 1)
Family A12:1b (transpose of A12:1a, self-conjugate)	
Cyclic rank 2 (suffices a and b)	
$\begin{pmatrix} 2a+2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2a & 1 & 1 & 1 & 1 \\ -1 & -1 & 2a+1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2a+1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2a+1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2a+1 \end{pmatrix}$	Order, n (1 6 25 60 72 40 8) /2 n_a (0 1 5 20 40 32 8) /2 n_b (0 0 0 0 0 1 1) g_{1a} (0 0 3 4 21 32 12) /2 g_{1b} (0 0 0 0 0 0 0) g_{2a} (0 0 0 5 31 46 20) /2 g_{2b} (0 0 0 0 0 0 0) g_{3a} (0 0 1 3 -12 -16 -2) /2 g_{3b} (0 0 0 0 0 0 0) g_{4a} (0 0 0 2 -1 -10 -5) g_{4b} (0 0 0 0 0 1 0) g_{5a} (0 0 1 6 15 14 3) g_{5b} (0 0 0 0 0 0 0) g_{6a} (0 0 1 6 15 14 3) g_{6b} (0 0 0 0 0 0 1)

Table C.12: Degree 12, diameter class 2 (mod 3), $a = (k + 1)/3$

LGM odd basis						Polynomial in $2a$						
Family A12:2 (self-transpose, conjugate of A12:0)												
Cyclic rank 2 (suffices a and b)												
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a & 0 & -1 & -1 \\ 1 & 1 & 0 & 2a & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 \\ 0 & 1 & 1 & 1 & 0 & 2a-1 \end{pmatrix}$	Order, n	(1	-2	12	-16	10	0	0)	/2			
	n_a	(0	1	-2	12	-16	10	0)	/2			
	n_b	(0	0	0	0	0	1	0)				
	g_{1a}	(0	0	2	0	19	-17	0)	/2			
	g_{1b}	(0	0	0	0	0	0	0)				
	g_{2a}	(0	0	0	4	-8	13	2)	/2			
	g_{2b}	(0	0	0	0	0	0	0)				
	g_{3a}	(0	0	0	1	-12	14	-6)	/2			
	g_{3b}	(0	0	0	0	0	1	-1)				
	g_{4a}	(0	0	1	-1	0	-2	4)	/2			
	g_{4b}	(0	0	0	0	0	0	1)				
	g_{5a}	(0	0	1	-5	8	-15	2)	/2			
	g_{5b}	(0	0	0	0	0	0	0)				
	g_{6a}	(0	0	2	-3	17	5	0)	/2			
g_{6b}	(0	0	0	0	0	0	0)					

Table C.13: Degree 13, diameter class 0 (mod 3), $a = k/3$

LGM	Polynomial in $2a$							
Family A13:0a (transpose of A13:0b, self-conjugate)								
For diameter $k \equiv 0 \pmod{3}$, except $k \equiv 138 \pmod{669}$ (gcd 223)								
Cyclic rank 3 (suffices a , b and c)								
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 2a & -1 & -1 & -1 & 0 \\ 1 & 1 & 2a & 0 & -1 & -1 \\ 1 & 1 & 0 & 2a & -1 & -1 \\ 0 & 1 & 1 & 1 & 2a & -1 \\ 0 & 0 & 1 & 1 & 1 & 2a-1 \end{pmatrix}$	Order, n	(1	0	10	0	0	0	0)
	n_a	(0	0	1	0	10	0	0)
	n_b	(0	0	0	0	0	1	0)
	n_c	(0	0	0	0	0	1	0)
	g_{1a}	(0	0	0	1	2	3	-5)
	g_{1b}	(0	0	0	0	0	0	1)
	g_{1c}	(0	0	0	0	0	1	-1)
	g_{2a}	(0	0	0	1	1	-2	-1)
	g_{2b}	(0	0	0	0	0	0	1)
	g_{2c}	(0	0	0	0	0	0	0)
	g_{3a}	(0	0	0	1	-3	0	-2)
	g_{3b}	(0	0	0	0	0	0	1)
	g_{3c}	(0	0	0	0	0	0	0)
	g_{4a}	(0	0	0	1	-3	0	-2)
	g_{4b}	(0	0	0	0	0	1	-1)
	g_{4c}	(0	0	0	0	0	1	-1)
	g_{5a}	(0	0	0	0	4	-2	1)
	g_{5b}	(0	0	0	0	0	1	-1)
	g_{5c}	(0	0	0	0	0	0	0)
	g_{6a}	(0	0	0	1	3	3	5)
	g_{6b}	(0	0	0	0	0	1	-1)
	g_{6c}	(0	0	0	0	0	0	1)
	g_{ma}	(0	0	1	0	10	0	0) /2
	g_{mb}	(0	0	0	0	0	0	0)
	g_{mc}	(0	0	0	0	0	1	0) /2

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Table C.13: (cont.) Degree 13, diameter class 0 (mod 3), $a = k/3$

LGM	Polynomial in $2a$
Family A13:0b (transpose of A13:0a, self-conjugate)	
For diameter $k \equiv 0 \pmod{3}$, except $k \equiv 99 \pmod{111}$ (gcd 37)	
Cyclic rank 3 (suffices a , b and c)	
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2a & 1 & 1 & 1 & 0 \\ -1 & -1 & 2a & 0 & 1 & 1 \\ -1 & -1 & 0 & 2a & 1 & 1 \\ 0 & -1 & -1 & -1 & 2a & 1 \\ 0 & 0 & -1 & -1 & -1 & 2a-1 \end{pmatrix}$	Order, n (1 0 10 0 0 0 0) n_a (0 0 1 0 10 0 0) n_b (0 0 0 0 0 1 0) n_c (0 0 0 0 0 1 0) g_{1a} (0 0 0 1 1 4 -1) g_{1b} (0 0 0 0 0 0 1) g_{1c} (0 0 0 0 0 0 0) g_{2a} (0 0 0 1 -3 9 -1) g_{2b} (0 0 0 0 0 0 1) g_{2c} (0 0 0 0 0 1 -1) g_{3a} (0 0 0 0 1 8 0) g_{3b} (0 0 0 0 0 1 -1) g_{3c} (0 0 0 0 0 0 0) g_{4a} (0 0 0 1 -1 2 0) g_{4b} (0 0 0 0 0 1 -1) g_{4c} (0 0 0 0 0 0 1) g_{5a} (0 0 0 1 2 3 1) g_{5b} (0 0 0 0 0 1 -1) g_{5c} (0 0 0 0 0 1 -1) g_{6a} (0 0 0 0 2 2 -1) g_{6b} (0 0 0 0 0 0 1) g_{6c} (0 0 0 0 0 0 0) g_{ma} (0 0 0 0 0 0 0) g_{mb} (0 0 0 0 0 0 0) g_{mc} (0 0 0 0 0 1 0) /2
Family A13:0c (self-transpose, self-conjugate)	
For diameter $k \equiv 0 \pmod{6}$	
Cyclic rank 4 (suffices a , b , c and d)	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a & 0 & 1 & -1 \\ 1 & 1 & 0 & 2a & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & -1 \\ 0 & 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order, n (1 0 12 0 0 0 0) n_a (0 0 0 1 0 12 0) /2 n_b (0 0 0 0 0 2 0) n_c (0 0 0 0 0 1 0) n_d (0 0 0 0 0 1 0) No formulae discovered yet for a generating set
Family A13:0d (self-transpose, self-conjugate)	
For diameter $k \equiv 3 \pmod{6}$	
Cyclic rank 4 (suffices a , b , c and d)	
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & 0 \\ 1 & 2a & -1 & -1 & 0 & -1 \\ 1 & 1 & 2a & 0 & 1 & -1 \\ 1 & 1 & 0 & 2a & 1 & -1 \\ 1 & 0 & -1 & -1 & 2a & -1 \\ 0 & 1 & 1 & 1 & 1 & 2a \end{pmatrix}$	Order, n (1 0 12 0 0 0 0) n_a (0 0 0 1 0 12 0) /4 n_b (0 0 0 0 0 4 0) n_c (0 0 0 0 0 1 0) n_d (0 0 0 0 0 1 0) No formulae discovered yet for a generating set

Table C.14: Degree 13, diameter class 1 (mod 3), $a = (k - 1)/3$

LGM						Polynomial in $2a$							
A13:1a1 (transpose of A13:1b1, conjugate of A13:2a)													
For diameter $k \equiv 1 \pmod{6}$, except $k \equiv 487 \pmod{606}$ (gcd 101)													
Cyclic rank 2 (suffices a and b)													
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2a+1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2a+1 \end{pmatrix}$						Order, n	(1	4	17	34	28	8	0)
						n_a	(0	1	3	14	20	8	0) /4
						n_b	(0	0	0	0	0	4	4)
						g_{1a}	(0	1	7	10	12	24	16) /16
						g_{1b}	(0	0	0	0	0	1	1)
						g_{2a}	(0	1	3	-2	-44	-56	-16) /16
						g_{2b}	(0	0	0	0	0	1	1)
						g_{3a}	(0	1	3	-6	16	24	0) /16
						g_{3b}	(0	0	0	0	0	3	3)
						g_{4a}	(0	1	-1	2	16	8	0) /16
						g_{4b}	(0	0	0	0	0	1	-1)
						g_{5a}	(0	1	-1	2	16	8	0) /16
						g_{5b}	(0	0	0	0	0	1	-5)
						g_{6a}	(0	1	-5	-6	-32	-24	0) /16
						g_{6b}	(0	0	0	0	0	3	11)
						g_{ma}	(0	0	0	0	0	0	0)
						g_{mb}	(0	0	0	0	0	2	2)
Family A13:1a4 (transpose of A13:1b4, conjugate of A13:2a)													
For diameter $k \equiv 4 \pmod{6}$, except $k \equiv 106 \pmod{186}$ (gcd 31)													
Cyclic rank 2 (suffices a and b)													
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2a+1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2a+1 \end{pmatrix}$						Order, n	(1	4	17	34	28	8	0)
						n_a	(0	1	3	14	20	8	0) /4
						n_b	(0	0	0	0	0	4	4)
						g_{1a}	(0	1	9	24	40	40	16) /16
						g_{1b}	(0	0	0	0	0	1	1)
						g_{2a}	(0	1	5	12	-16	-40	-16) /16
						g_{2b}	(0	0	0	0	0	1	1)
						g_{3a}	(0	1	5	0	4	8	0) /16
						g_{3b}	(0	0	0	0	0	0	0)
						g_{4a}	(0	1	1	0	12	8	0) /16
						g_{4b}	(0	0	0	0	0	0	2)
						g_{5a}	(0	1	1	0	12	8	0) /16
						g_{5b}	(0	0	0	0	0	3	4)
						g_{6a}	(0	1	-3	-8	-36	-24	0) /16
						g_{6b}	(0	0	0	0	0	3	0)
						g_{ma}	(0	0	0	0	0	0	0)
						g_{mb}	(0	0	0	0	0	2	2)

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Table C.14: (cont.) Degree 13, diameter class 1 (mod 3), $a = (k - 1)/3$

LGM						Polynomial in $2a$								
Family A13:1b1 (transpose of A13:1a1, conjugate of A13:2b)														
For diameter $k \equiv 1 \pmod{6}$, except $k \equiv 349 \pmod{402}$ (gcd 67)														
Cyclic rank 2 (suffices a and b)														
$\begin{pmatrix} 2a & 0 & 1 & 1 & 1 & 1 \\ 0 & 2a & 1 & 1 & 1 & 1 \\ -1 & -1 & 2a+1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2a+1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2a+1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2a+1 \end{pmatrix}$						Order, n	(1	4	17	34	28	8	0)	
						n_a	(0	1	3	14	20	8	0)	/4
						n_b	(0	0	0	0	0	4	4)	
						g_{1a}	(0	1	-1	6	24	32	16)	/16
						g_{1b}	(0	0	0	0	0	1	1)	
						g_{2a}	(0	1	-5	-6	-32	-48	-16)	/16
						g_{2b}	(0	0	0	0	0	3	3)	
						g_{3a}	(0	1	1	0	24	24	0)	/16
						g_{3b}	(0	0	0	0	0	1	1)	
						g_{4a}	(0	1	5	4	16	8	0)	/16
						g_{4b}	(0	0	0	0	0	1	-11)	
						g_{5a}	(0	1	5	4	16	8	0)	/16
						g_{5b}	(0	0	0	0	0	3	13)	
						g_{6a}	(0	1	-3	-12	-80	-56	0)	/16
						g_{6b}	(0	0	0	0	0	3	5)	
						g_{ma}	(0	1	3	14	20	8	0)	/8
						g_{mb}	(0	0	0	0	0	2	2)	
Family A13:1b4 (transpose of A13:1a4, conjugate of A13:2b)														
For diameter $k \equiv 4 \pmod{6}$, except $k \equiv 118 \pmod{174}$ (gcd 29)														
Cyclic rank 2 (suffices a and b)														
$\begin{pmatrix} 2a & 0 & 1 & 1 & 1 & 1 \\ 0 & 2a & 1 & 1 & 1 & 1 \\ -1 & -1 & 2a+1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2a+1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2a+1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2a+1 \end{pmatrix}$						Order, n	(1	4	17	34	28	8	0)	
						n_a	(0	1	3	14	20	8	0)	/4
						n_b	(0	0	0	0	0	4	4)	
						g_{1a}	(0	1	1	4	20	32	16)	/16
						g_{1b}	(0	0	0	0	0	1	1)	
						g_{2a}	(0	1	-3	-8	-36	-48	-16)	/16
						g_{2b}	(0	0	0	0	0	3	3)	
						g_{3a}	(0	1	3	2	16	16	0)	/16
						g_{3b}	(0	0	0	0	0	1	1)	
						g_{4a}	(0	1	7	14	32	16	0)	/16
						g_{4b}	(0	0	0	0	0	1	3)	
						g_{5a}	(0	1	7	14	32	16	0)	/16
						g_{5b}	(0	0	0	0	0	1	-1)	
						g_{6a}	(0	1	-1	-2	-64	-48	0)	/16
						g_{6b}	(0	0	0	0	0	3	3)	
						g_{ma}	(0	1	3	14	20	8	0)	/8
						g_{mb}	(0	0	0	0	0	0	0)	

Table C.15: Degree 13, diameter class 2 (mod 3), $a = (k + 1)/3$

LGM						Polynomial in $2a$							
Family A13:2a2 (transpose of A13:2b2, conjugate of A13:1a)													
For diameter $k \equiv 2 \pmod{6}$, except $k \equiv 170 \pmod{222}$ (gcd 37)													
Cyclic rank 2 (suffices a and b)													
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a-1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a-1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2a-1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2a-1 \end{pmatrix}$						Order, n	(1	-4	17	-34	28	-8	0)
						n_a	(0	1	-3	14	-20	8	0) /4
						n_b	(0	0	0	0	0	4	-4)
						g_{1a}	(0	1	3	-8	20	-32	16) /16
						g_{1b}	(0	0	0	0	0	1	-1)
						g_{2a}	(0	1	-1	4	-36	48	-16) /16
						g_{2b}	(0	0	0	0	0	3	-3)
						g_{3a}	(0	1	1	-10	24	-16	0) /16
						g_{3b}	(0	0	0	0	0	1	-1)
						g_{4a}	(0	1	-3	2	24	-16	0) /16
						g_{4b}	(0	0	0	0	0	3	1)
						g_{5a}	(0	1	-3	2	24	-16	0) /16
						g_{5b}	(0	0	0	0	0	3	-3)
						g_{6a}	(0	1	-11	18	-72	48	0) /16
						g_{6b}	(0	0	0	0	0	3	-7)
						g_{ma}	(0	1	-3	14	-20	8	0) /8
						g_{mb}	(0	0	0	0	0	0	0)
Family A13:2a5 (transpose of A13:2b5, conjugate of A13:1a)													
For diameter $k \equiv 5 \pmod{6}$, except $k \equiv 47 \pmod{78}$ (gcd 13)													
Cyclic rank 2 (suffices a and b)													
$\begin{pmatrix} 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2a+1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2a+1 \end{pmatrix}$						Order, n	(1	-4	17	-34	28	-8	0)
						n_a	(0	1	-3	14	-20	8	0) /4
						n_b	(0	0	0	0	0	4	-4)
						g_{1a}	(0	1	1	-6	20	-32	16) /16
						g_{1b}	(0	0	0	0	0	1	-1)
						g_{2a}	(0	1	-3	6	-36	48	-16) /16
						g_{2b}	(0	0	0	0	0	3	-3)
						g_{3a}	(0	1	-1	-4	20	-16	0) /16
						g_{3b}	(0	0	0	0	0	1	-1)
						g_{4a}	(0	1	-5	12	-4	0	0) /16
						g_{4b}	(0	0	0	0	0	1	-1)
						g_{5a}	(0	1	-5	12	-4	0	0) /16
						g_{5b}	(0	0	0	0	0	3	-1)
						g_{6a}	(0	1	-9	20	-52	32	0) /16
						g_{6b}	(0	0	0	0	0	3	-5)
						g_{ma}	(0	0	0	0	0	0	0) /8
						g_{mb}	(0	0	0	0	0	2	-2)

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Table C.15: (cont.) Degree 13, diameter class 2 (mod 3), $a = (k + 1)/3$

LGM	Polynomial in $2a$
Family A13:2b2 (transpose of A13:2a2, conjugate of A13:1b)	
For diameter $k \equiv 2 \pmod{6}$, except $k \equiv 80 \pmod{186}$ (gcd 31)	
Cyclic rank 2 (suffices a and b)	
$\begin{pmatrix} 2a & 0 & 1 & 1 & 1 & 1 \\ 0 & 2a & 1 & 1 & 1 & 1 \\ -1 & -1 & 2a-1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2a-1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2a-1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2a-1 \end{pmatrix}$	Order, n (1 -4 17 -34 28 -8 0) n_a (0 1 -3 14 -20 8 0) /4 n_b (0 0 0 0 0 4 -4) g_{1a} (0 1 -9 24 -40 40 -16) /16 g_{1b} (0 0 0 0 0 1 -1) g_{2a} (0 1 -5 12 16 -40 16) /16 g_{2b} (0 0 0 0 0 1 -1) g_{3a} (0 1 -5 0 -4 8 0) /16 g_{3b} (0 0 0 0 0 0 0) g_{4a} (0 1 -1 0 -12 8 0) /16 g_{4b} (0 0 0 0 0 4 -6) g_{5a} (0 1 -1 0 -12 8 0) /16 g_{5b} (0 0 0 0 0 3 -4) g_{6a} (0 1 3 -8 36 -24 0) /16 g_{6b} (0 0 0 0 0 3 0) g_{ma} (0 0 0 0 0 0 0) g_{mb} (0 0 0 0 0 2 -2)
Family A13:2b5 (transpose of A13:2a5, conjugate of A13:1b)	
For diameter $k \equiv 5 \pmod{6}$, except $k \equiv 119 \pmod{606}$ (gcd 101)	
Cyclic rank 2 (suffices a and b)	
$\begin{pmatrix} 2a & 0 & 1 & 1 & 1 & 1 \\ 0 & 2a & 1 & 1 & 1 & 1 \\ -1 & -1 & 2a-1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 2a-1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2a-1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2a-1 \end{pmatrix}$	Order, n (1 -4 17 -34 28 -8 0) n_a (0 1 -3 14 -20 8 0) /4 n_b (0 0 0 0 0 4 -4) g_{1a} (0 1 -7 10 -12 24 -16) /16 g_{1b} (0 0 0 0 0 1 -1) g_{2a} (0 1 -3 -2 44 -56 16) /16 g_{2b} (0 0 0 0 0 1 -1) g_{3a} (0 1 -3 -6 -16 24 0) /16 g_{3b} (0 0 0 0 0 3 -3) g_{4a} (0 1 1 2 -16 8 0) /16 g_{4b} (0 0 0 0 0 1 1) g_{5a} (0 1 1 2 -16 8 0) /16 g_{5b} (0 0 0 0 0 1 5) g_{6a} (0 1 5 -6 32 -24 0) /16 g_{6b} (0 0 0 0 0 3 -11) g_{ma} (0 0 0 0 0 0 0) g_{mb} (0 0 0 0 0 2 -2)

C.5 Abelian Cayley graph families of degrees 14 and 15

Table C.16: Degree 14, diameter class 0 (mod 7), except $k \equiv 1064 \pmod{1211}$
(gcd 173), $a = 2k/7$

LGM odd basis							Polynomial in $2a$									
Family A14:0 (self-transpose, conjugate of A14:6, translate of A15:4)																
Cyclic rank 2 (suffices a and b)																
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n	(1	2	16	20	13	6	0	0)	/2
							n_a	(0	1	2	16	20	13	6	0)	/2
							n_b	(0	0	0	0	0	0	1	0)	
							g_{1a}	(0	0	1	0	7	5	2	0)	/2
							g_{1b}	(0	0	0	0	0	0	0	0)	
							g_{2a}	(0	0	0	2	2	-1	2	0)	/2
							g_{2b}	(0	0	0	0	0	0	0	0)	
							g_{3a}	(0	0	0	1	-5	-7	-5	-2)	/2
							g_{3b}	(0	0	0	0	0	0	1	-1)	
							g_{4a}	(0	0	0	1	-5	-7	-5	-2)	/2
							g_{4b}	(0	0	0	0	0	0	0	0)	
							g_{5a}	(0	0	1	3	11	13	8	4)	/2
							g_{5b}	(0	0	0	0	0	0	0	1)	
							g_{6a}	(0	0	0	2	5	8	4	0)	/2
							g_{6b}	(0	0	0	0	0	0	0	0)	
							g_{7a}	(0	0	0	1	7	3	-2	0)	/2
							g_{7b}	(0	0	0	0	0	0	0	0)	

Table C.17: Degree 14, diameter class 1 (mod 7), $a = (2k - 2)/7$

Family A14:1a (transpose of A14:1b, conjugate of A14:5a, translate of A15:5a) Cyclic rank 2 (suffices a and b)									
LGM odd basis									
$\begin{pmatrix} 2a+1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 2a+1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 2a+1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 2a+1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 2a+1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 2a+1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 2a \end{pmatrix}$									
Polynomial in $2a$									
Order, n	(1	6	29	80	125	112	53	10)	/2
n_a	(0	1	5	24	56	69	43	10)	/2
n_b	(0	0	0	0	0	0	1	1)	
g_{1a}	(0	0	1	4	15	25	22	7)	
g_{1b}	(0	0	0	0	0	0	0	1)	
g_{2a}	(0	0	0	1	0	-4	-6	-3)	
g_{2b}	(0	0	0	0	0	0	0	0)	
g_{3a}	(0	0	0	1	4	6	4	1)	
g_{3b}	(0	0	0	0	0	0	0	1)	
g_{4a}	(0	0	0	1	4	6	4	1)	
g_{4b}	(0	0	0	0	0	0	1	0)	
g_{5a}	(0	0	0	1	2	2	0	-1)	
g_{5b}	(0	0	0	0	0	0	0	0)	
g_{6a}	(0	0	0	0	5	11	11	3)	
g_{6b}	(0	0	0	0	0	0	0	1)	
g_{7a}	(0	0	0	1	6	12	12	5)	
g_{7b}	(0	0	0	0	0	0	0	0)	

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Table C.17: (cont.) Degree 14, diameter class 1 (mod 7), $a = (2k - 2)/7$

Family A14:1b (transpose of A14:1a, conjugate of A14:5b, translate of A15:5b) Cyclic rank 2 (suffices a and b)									
LGM odd basis									
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & 2a+1 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 2a+1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2a+1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 2a+1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 2a+1 & -1 \\ -1 & -1 & 0 & 0 & -1 & 1 & 2a \end{pmatrix}$									
Polynomial in $2a$									
Order, n	(1	6	29	80	125	112	53	10)	/2
n_a	(0	1	5	24	56	69	43	10)	/2
n_b	(0	0	0	0	0	0	1	1)	
g_{1a}	(0	0	1	6	21	31	24	7)	
g_{1b}	(0	0	0	0	0	0	1	0)	
g_{2a}	(0	0	1	5	14	30	27	7)	
g_{2b}	(0	0	0	0	0	0	0	0)	
g_{3a}	(0	0	1	2	9	11	2	-1)	
g_{3b}	(0	0	0	0	0	0	1	0)	
g_{4a}	(0	0	1	2	9	11	2	-1)	
g_{4b}	(0	0	0	0	0	0	0	1)	
g_{5a}	(0	0	1	5	28	58	45	11)	
g_{5b}	(0	0	0	0	0	0	0	0)	
g_{6a}	(0	0	0	2	1	-5	-9	-3)	
g_{6b}	(0	0	0	0	0	0	0	1)	
g_{7a}	(0	0	0	1	8	8	-2	-3)	
g_{7b}	(0	0	0	0	0	0	0	0)	
Family A14:1c (self-transpose, conjugate of A14:5c, translate of A15:5c) Cyclic rank 2 (suffices a and b)									
LGM odd basis									
$\begin{pmatrix} 2a+2 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a+1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 2a+1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2a+1 \end{pmatrix}$									
Polynomial in $2a$									
Order, n	(1	6	30	84	113	70	16	0)	/2
n_a	(0	1	5	25	59	54	16	0)	/8
n_b	(0	0	0	0	0	0	4	4)	
No formulae discovered yet for a generating set									

Table C.18: Degree 14, diameter class 2 (mod 7), $a = (2k + 3)/7$

LGM odd basis	Polynomial in $2a$
Family A14:2 (self-transpose, conjugate of A14:4, translate of A15:6)	
Cyclic rank 2 (suffices a and b)	
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a-1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a-1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a-1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$	Order, n (1 -4 22 -48 41 -12 0 0) /2
	n_a (0 0 1 -3 19 -29 12 0) /2
	n_b (0 0 0 0 0 1 -1 0)
	g_{1a} (0 0 0 1 -3 12 -8 0) /2
	g_{1b} (0 0 0 0 0 0 0 0)
	g_{2a} (0 0 0 0 1 5 -4 0) /2
	g_{2b} (0 0 0 0 0 0 0 0)
	g_{3a} (0 0 0 0 1 -1 0 0)
	g_{3b} (0 0 0 0 0 1 -2 0)
	g_{4a} (0 0 0 0 1 -1 0 0)
	g_{4b} (0 0 0 0 0 0 1 0)
	g_{5a} (0 0 0 0 1 -8 11 -4) /2
	g_{5b} (0 0 0 0 0 1 -2 1)
	g_{6a} (0 0 0 0 1 -8 11 -4) /2
	g_{6b} (0 0 0 0 0 0 0 0)
	g_{7a} (0 0 0 1 -2 11 -18 8) /2
	g_{7b} (0 0 0 0 0 0 1 -1)

Table C.19: Degree 14, diameter class 3 (mod 7), except $k \equiv 283 \pmod{511}$
(gcd 73), $a = (2k + 1)/7$

LGM odd basis	Polynomial in $2a$
Family A14:3 (self-transpose?)	
Cyclic rank 2 (suffices a and b)	
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 2a+1 \end{pmatrix}$	Order, n (1 0 13 2 7 -2 0 0) /2
	n_a (0 1 0 13 2 7 -2 0)
	n_b (0 0 0 0 0 0 1 0) /2
	g_{1a} (0 0 1 -3 0 -2 1 0)
	g_{1b} (0 0 0 0 0 0 1 -2) /2
	g_{2a} (0 0 1 -3 2 -2 0 0)
	g_{2b} (0 0 0 0 0 0 0 1)
	g_{3a} (0 0 1 1 -1 3 -1 0)
	g_{3b} (0 0 0 0 0 0 0 1)
	g_{4a} (0 0 1 4 7 3 3 -1)
	g_{4b} (0 0 0 0 0 0 1 -2) /2
	g_{5a} (0 0 0 4 -6 1 -4 1)
	g_{5b} (0 0 0 0 0 0 0 1)
	g_{6a} (0 0 1 -1 0 -3 1 0)
	g_{6b} (0 0 0 0 0 0 1 -2) /2
	g_{7a} (0 0 1 2 4 0 0 0)
	g_{7b} (0 0 0 0 0 0 1 -2) /2

Table C.20: Degree 14, diameter class 4 (mod 7), $a = (2k - 1)/7$

LGM odd basis							Polynomial in $2a$									
Family A14:4 (self-transpose, conjugate of A14:2, translate of A15:1)																
Cyclic rank 2 (suffices a and b)																
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a+1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n (1 4 22 48 41 12 0 0) /2									
							n_a	(0 0 1 3 19 29 12 0) /2								
							n_b	(0 0 0 0 0 1 1 0)								
							g_{1a}	(0 0 0 1 3 12 8 0) /2								
							g_{1b}	(0 0 0 0 0 0 0 0)								
							g_{2a}	(0 0 0 0 1 -5 -4 0) /2								
							g_{2b}	(0 0 0 0 0 0 0 0)								
							g_{3a}	(0 0 0 0 1 1 0 0)								
							g_{3b}	(0 0 0 0 0 1 0 0)								
							g_{4a}	(0 0 0 0 1 1 0 0)								
							g_{4b}	(0 0 0 0 0 0 1 0)								
							g_{5a}	(0 0 0 0 1 8 11 4) /2								
							g_{5b}	(0 0 0 0 0 0 1 1)								
							g_{6a}	(0 0 0 0 1 8 11 4) /2								
							g_{6b}	(0 0 0 0 0 0 0 0)								
							g_{7a}	(0 0 0 1 2 11 18 8) /2								
							g_{7b}	(0 0 0 0 0 0 1 1)								

Table C.21: Degree 14, diameter class 5 (mod 7), $a = (2k + 4)/7$

Family A14:5a (transpose of A14:5b, conjugate of A14:1a, translate of A15:2a) Cyclic rank 2 (suffices a and b)									
LGM odd basis									
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 2a-1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 2a-1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 2a-1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 2a-1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 2a \end{pmatrix}$									
Polynomial in $2a$									
Order, n	(1	-6	29	-80	125	-112	53	-10)	/2
n_a	(0	1	-5	24	-56	69	-43	10)	/2
n_b	(0	0	0	0	0	0	1	-1)	
g_{1a}	(0	0	1	-4	15	-25	22	-7)	
g_{1b}	(0	0	0	0	0	0	0	1)	
g_{2a}	(0	0	0	1	0	-4	6	-3)	
g_{2b}	(0	0	0	0	0	0	0	0)	
g_{3a}	(0	0	0	1	-4	6	-4	1)	
g_{3b}	(0	0	0	0	0	0	1	-2)	
g_{4a}	(0	0	0	1	-4	6	-4	1)	
g_{4b}	(0	0	0	0	0	0	0	1)	
g_{5a}	(0	0	0	1	-2	2	0	-1)	
g_{5b}	(0	0	0	0	0	0	0	0)	
g_{6a}	(0	0	0	0	5	-11	11	-3)	
g_{6b}	(0	0	0	0	0	0	0	1)	
g_{7a}	(0	0	0	1	-6	12	-12	5)	
g_{7b}	(0	0	0	0	0	0	0	0)	

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Table C.21: (cont.) Degree 14, diameter class 5 (mod 7), $a = (2k + 4)/7$

Family A14:5b (transpose of A14:5a, conjugate of A14:1b, translate of A15:2b) Cyclic rank 2 (suffices a and b)									
LGM odd basis									
$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & 2a-1 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 2a-1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2a-1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 2a-1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 2a-1 & -1 \\ -1 & -1 & 0 & 0 & -1 & 1 & 2a \end{pmatrix}$									
Polynomial in $2a$									
Order, n	(1	-6	29	-80	125	-112	53	-10)	/2
n_a	(0	1	-5	24	-56	69	-43	10)	/2
n_b	(0	0	0	0	0	0	1	-1)	
g_{1a}	(0	0	1	-6	21	-31	24	-7)	
g_{1b}	(0	0	0	0	0	0	0	1)	
g_{2a}	(0	0	1	-5	14	-30	27	-7)	
g_{2b}	(0	0	0	0	0	0	0	0)	
g_{3a}	(0	0	1	-2	9	-11	2	1)	
g_{3b}	(0	0	0	0	0	0	0	1)	
g_{4a}	(0	0	1	-2	9	-11	2	1)	
g_{4b}	(0	0	0	0	0	0	1	-2)	
g_{5a}	(0	0	1	-5	28	-58	45	-11)	
g_{5b}	(0	0	0	0	0	0	0	0)	
g_{6a}	(0	0	0	2	-1	-5	9	-3)	
g_{6b}	(0	0	0	0	0	0	0	1)	
g_{7a}	(0	0	0	1	-8	8	2	-3)	
g_{7b}	(0	0	0	0	0	0	0	0)	
Family A14:5c (self-transpose, conjugate of A14:1c, translate of A15:2c) Cyclic rank 2 (suffices a and b)									
LGM odd basis									
$\begin{pmatrix} 2a-2 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a-1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a-1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 2a-1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2a-1 \end{pmatrix}$									
Polynomial in $2a$									
Order, n	(1	-6	30	-84	113	-70	16	0)	/2
n_a	(0	1	-5	25	-59	54	-16	0)	/8
n_b	(0	0	0	0	0	0	4	-4)	
No formulae discovered yet for a generating set									

Table C.22: Degree 14, diameter class 6 (mod 7), $a = (2k + 2)/7$

LGM odd basis							Polynomial in $2a$											
Family A14:6 (self-transpose, conjugate of A14:0, translate of A15:3)																		
Cyclic rank 2 (suffices a and b)																		
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a-1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n	(1	-2	16	-20	13	-6	0	0)	/2		
							n_a	(0	1	-2	16	-20	13	-6	0)	/2		
							n_b	(0	0	0	0	0	0	1	0)			
							g_{1a}	(0	0	1	0	7	-5	2	0)	/2		
							g_{1b}	(0	0	0	0	0	0	0	0)			
							g_{2a}	(0	0	0	2	-2	-1	-2	0)	/2		
							g_{2b}	(0	0	0	0	0	0	0	0)			
							g_{3a}	(0	0	0	1	5	-7	5	-2)	/2		
							g_{3b}	(0	0	0	0	0	0	0	1)			
							g_{4a}	(0	0	0	1	5	-7	5	-2)	/2		
							g_{4b}	(0	0	0	0	0	0	0	0)			
							g_{5a}	(0	0	1	-3	11	-13	8	-4)	/2		
							g_{5b}	(0	0	0	0	0	0	0	1)			
							g_{6a}	(0	0	0	2	-5	8	-4	0)	/2		
							g_{6b}	(0	0	0	0	0	0	0	0)			
							g_{7a}	(0	0	0	1	-7	3	2	0)	/2		
							g_{7b}	(0	0	0	0	0	0	0	0)			

Table C.23: Degree 15, diameter class 0 (mod 7), except $k \equiv 63 \pmod{133}$
(gcd 19), $a = 2k/7$

LGM							Polynomial in $2a$									
Family A15:0 (self-transpose, self-conjugate)																
Cyclic rank 5 (suffices a, b, c, d and e)																
$\begin{pmatrix} 2a & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 2a & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 2a & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n	(1	0	15	0	0	0	0	0	0)
							n_a	(0	0	0	0	1	0	15	0)	
							n_b	(0	0	0	0	0	0	1	0)	
							n_c	(0	0	0	0	0	0	1	0)	
							n_d	(0	0	0	0	0	0	1	0)	
							n_e	(0	0	0	0	0	0	1	0)	
							g_{1a}	(0	0	0	0	0	1	5	6)	
							g_{1b}	(0	0	0	0	0	0	0	1)	
							g_{1c}	(0	0	0	0	0	0	0	1)	
							g_{1d}	(0	0	0	0	0	0	0	1)	
							g_{1e}	(0	0	0	0	0	0	0	1)	
							g_{2a}	(0	0	0	0	0	1	1	-2)	
							g_{2b}	(0	0	0	0	0	0	0	1)	
							g_{2c}	(0	0	0	0	0	0	0	1)	
							g_{2d}	(0	0	0	0	0	0	0	0)	
							g_{2e}	(0	0	0	0	0	0	0	0)	
							g_{3a}	(0	0	0	0	0	1	1	-2)	
							g_{3b}	(0	0	0	0	0	0	1	-1)	
							g_{3c}	(0	0	0	0	0	0	1	-1)	
							g_{3d}	(0	0	0	0	0	0	0	0)	
							g_{3e}	(0	0	0	0	0	0	1	-1)	
							g_{4a}	(0	0	0	0	0	1	1	-2)	
							g_{4b}	(0	0	0	0	0	0	1	-1)	
							g_{4c}	(0	0	0	0	0	0	1	-1)	
							g_{4d}	(0	0	0	0	0	0	1	-1)	
							g_{4e}	(0	0	0	0	0	0	0	0)	
							g_{5a}	(0	0	0	0	0	1	-4	7)	
							g_{5b}	(0	0	0	0	0	0	0	1)	
							g_{5c}	(0	0	0	0	0	0	0	1)	
							g_{5d}	(0	0	0	0	0	0	0	1)	
							g_{5e}	(0	0	0	0	0	0	0	1)	
							g_{6a}	(0	0	0	0	0	1	-4	7)	
							g_{6b}	(0	0	0	0	0	0	0	1)	
							g_{6c}	(0	0	0	0	0	0	0	0)	
							g_{6d}	(0	0	0	0	0	0	0	1)	
							g_{6e}	(0	0	0	0	0	0	0	1)	
							g_{7a}	(0	0	0	0	1	0	11	-8)	
							g_{7b}	(0	0	0	0	0	0	1	-1)	
							g_{7c}	(0	0	0	0	0	0	0	0)	
							g_{7d}	(0	0	0	0	0	0	1	-1)	
g_{7e}	(0	0	0	0	0	0	1	-1)								
g_{ma}	(0	0	0	0	0	0	0	0)								
g_{mb}	(0	0	0	0	0	0	1	0) /2								
g_{mc}	(0	0	0	0	0	0	1	0) /2								
g_{md}	(0	0	0	0	0	0	1	0) /2								
g_{me}	(0	0	0	0	0	0	1	0) /2								

Table C.24: Degree 15, diameter class 1 (mod 7), $a = (2k - 2)/7$

LGM							Polynomial in $2a$	
Family A15:1 (self-transpose?, conjugate of A15:6, translate of A14:4)								
Cyclic rank 3 (suffices a , b and c)								
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a+1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a+1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n	(1 4 22 48 41 12 0 0)
							n_a	(0 1 3 19 29 12 0 0) /16
							n_b	(0 0 0 0 0 0 4 4)
							n_c	(0 0 0 0 0 0 0 4)
							No formulae discovered yet for a generating set	

Table C.25: Degree 15, diameter class 2 (mod 7), $a = (2k + 3)/7$

Family A15:2a (transpose of A15:2b, conjugate of A15:5a, translate of A14:5a)							
Cyclic rank 2 (suffices a and b)							
LGM							
$\begin{pmatrix} 2a-1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 2a-1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 2a-1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 2a-1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 2a-1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 2a-1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 2a \end{pmatrix}$							
Polynomial in $2a$							
Order, n	(1	-6	29	-80	125	-112	53 -10)
n_a	(0	1	-5	24	-56	69	-43 10)
n_b	(0	0	0	0	0	0	1 -1)
g_{1a}	(0	0	1	1	4	-14	19 -7)
g_{1b}	(0	0	0	0	0	0	0 1)
g_{2a}	(0	0	1	-1	-4	10	-9 3)
g_{2b}	(0	0	0	0	0	0	0 0)
g_{3a}	(0	0	1	-5	10	-10	5 -1)
g_{3b}	(0	0	0	0	0	0	0 1)
g_{4a}	(0	0	1	-5	10	-10	5 -1)
g_{4b}	(0	0	0	0	0	0	1 -2)
g_{5a}	(0	0	1	-3	4	-2	-1 1)
g_{5b}	(0	0	0	0	0	0	0 0)
g_{6a}	(0	0	1	-9	36	-58	47 -13)
g_{6b}	(0	0	0	0	0	0	0 1)
g_{7a}	(0	0	1	-7	18	-24	17 -5)
g_{7b}	(0	0	0	0	0	0	0 0)
g_{ma}	(0	1	-5	24	-56	69	-43 10) /2
g_{mb}	(0	0	0	0	0	0	0 0)

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Table C.25: (cont.) Degree 15, diameter class 2 (mod 7), $a = (2k + 3)/7$

Family A15:2b (transpose of A15:2a, conjugate of A15:5b, translate of A14:5b)								
Cyclic rank 2 (suffices a and b)								
LGM								
$\begin{pmatrix} 2a-1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & 2a-1 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 2a-1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2a-1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 2a-1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 2a-1 & -1 \\ -1 & -1 & 0 & 0 & -1 & 1 & 2a \end{pmatrix}$								
Polynomial in $2a$								
Order, n	(1	-6	29	-80	125	-112	53	-10)
n_a	(0	1	-5	24	-56	69	-43	10)
n_b	(0	0	0	0	0	0	1	-1)
g_{1a}	(0	0	1	-6	21	-31	24	-7)
g_{1b}	(0	0	0	0	0	0	0	1)
g_{2a}	(0	0	1	-5	14	-30	27	-7)
g_{2b}	(0	0	0	0	0	0	0	0)
g_{3a}	(0	0	1	-2	9	-11	2	1)
g_{3b}	(0	0	0	0	0	0	0	1)
g_{4a}	(0	0	1	-2	9	-11	2	1)
g_{4b}	(0	0	0	0	0	0	1	-2)
g_{5a}	(0	0	1	-5	28	-58	45	-11)
g_{5b}	(0	0	0	0	0	0	0	0)
g_{6a}	(0	0	0	2	-1	-5	9	-3)
g_{6b}	(0	0	0	0	0	0	0	1)
g_{7a}	(0	0	0	1	-8	8	2	-3)
g_{7b}	(0	0	0	0	0	0	0	0)
g_{ma}	(0	1	-5	24	-56	69	-43	10) /2
g_{mb}	(0	0	0	0	0	0	0	0)

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Table C.25: (cont.) Degree 15, diameter class 2 (mod 7), $a = (2k + 3)/7$

Family A15:2c (self-transpose, conjugate of A15:5c, translate of A14:5c)								
Cyclic rank 3 (suffices a , b and c)								
LGM								
$\begin{pmatrix} 2a-2 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a-1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a-1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 2a-1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2a-1 \end{pmatrix}$								
Polynomial in $2a$								
Order, n	(1	-6	30	-84	113	-70	16	0)
n_a	(0	1	-5	25	-59	54	-16	0) /4
n_b	(0	0	0	0	0	0	2	-2)
n_c	(0	0	0	0	0	0	0	2)
g_{1a}	(0	1	-9	43	-87	68	-16	0) /8
g_{1b}	(0	0	0	0	0	0	1	-1)
g_{1c}	(0	0	0	0	0	0	0	1)
g_{2a}	(0	1	-9	29	-49	40	-8	0) /8
g_{2b}	(0	0	0	0	0	0	1	0)
g_{2c}	(0	0	0	0	0	0	0	1)
g_{3a}	(0	1	-5	7	-7	12	-8	0) /8
g_{3b}	(0	0	0	0	0	0	1	0)
g_{3c}	(0	0	0	0	0	0	0	1)
g_{4a}	(0	1	-5	7	-7	12	-8	0) /8
g_{4b}	(0	0	0	0	0	0	1	-2)
g_{4c}	(0	0	0	0	0	0	0	1)
g_{5a}	(0	1	-7	21	-7	-36	24	0) /8
g_{5b}	(0	0	0	0	0	0	1	-2)
g_{5c}	(0	0	0	0	0	0	0	1)
g_{6a}	(0	1	-3	9	-45	102	-96	32) /8
g_{6b}	(0	0	0	0	0	0	0	0)
g_{6c}	(0	0	0	0	0	0	0	1)
g_{7a}	(0	1	1	-11	55	-134	120	-32) /8
g_{7b}	(0	0	0	0	0	0	1	-1)
g_{7c}	(0	0	0	0	0	0	0	1)
g_{ma}	(0	1	-5	25	-59	54	-16	0) /8
g_{mb}	(0	0	0	0	0	0	1	-1)
g_{mc}	(0	0	0	0	0	0	0	0)

Table C.26: Degree 15, diameter class 3 (mod 7), $a = (2k + 1)/7$

LGM							Polynomial in $2a$									
Family A15:3 (self-transpose, conjugate of A15:4, translate of A14:6)																
Cyclic rank 3 (suffices a , b and c)																
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a-1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n	(1	-2	16	-20	13	-6	0	0)	
							n_a	(0	1	-2	16	-20	13	-6	0)	/2
							n_b	(0	0	0	0	0	0	1	0)	
							n_c	(0	0	0	0	0	0	0	2)	
							g_{1a}	(0	1	-7	29	-41	26	-14	0)	/4
							g_{1b}	(0	0	0	0	0	0	1	0)	/2
							g_{1c}	(0	0	0	0	0	0	0	1)	
							g_{2a}	(0	1	-5	31	-39	26	-10	0)	/4
							g_{2b}	(0	0	0	0	0	0	1	0)	/2
							g_{2c}	(0	0	0	0	0	0	0	1)	
							g_{3a}	(0	1	-3	33	-39	22	-10	0)	/4
							g_{3b}	(0	0	0	0	0	0	1	0)	/2
							g_{3c}	(0	0	0	0	0	0	0	1)	
							g_{4a}	(0	1	-3	35	-35	26	-10	0)	/4
							g_{4b}	(0	0	0	0	0	0	1	0)	/2
							g_{4c}	(0	0	0	0	0	0	0	1)	
							g_{5a}	(0	1	2	18	-20	7	-2	-4)	/4
							g_{5b}	(0	0	0	0	0	0	1	0)	/2
							g_{5c}	(0	0	0	0	0	0	0	0)	
							g_{6a}	(0	1	2	18	-20	7	-2	-4)	/4
							g_{6b}	(0	0	0	0	0	0	1	2)	/2
							g_{6c}	(0	0	0	0	0	0	0	1)	
							g_{7a}	(0	0	1	3	-16	17	-11	4)	/2
							g_{7b}	(0	0	0	0	0	0	1	-2)	/2
							g_{7c}	(0	0	0	0	0	0	0	0)	
							g_{ma}	(0	1	-2	16	-20	13	-6	0)	/4
							g_{mb}	(0	0	0	0	0	0	0	0)	
							g_{mc}	(0	0	0	0	0	0	0	0)	

Table C.27: Degree 15, diameter class 4 (mod 7), $a = (2k - 1)/7$

LGM							Polynomial in $2a$										
Family A15:4 (self-transpose, conjugate of A15:3, translate of A14:0)																	
Cyclic rank 3 (suffices a , b and c)																	
$\begin{pmatrix} 2a+1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2a & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a+1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$							Order, n	(1	2	16	20	13	6	0	0)		
							n_a	(0	1	2	16	20	13	6	0)	/2	
							n_b	(0	0	0	0	0	0	0	1	0)	
							n_c	(0	0	0	0	0	0	0	0	2)	
							g_{1a}	(0	1	7	29	41	26	14	0)	/4	
							g_{1b}	(0	0	0	0	0	0	0	1	0)	/2
							g_{1c}	(0	0	0	0	0	0	0	0	1)	
							g_{2a}	(0	1	5	31	39	26	10	0)	/4	
							g_{2b}	(0	0	0	0	0	0	0	1	0)	/2
							g_{2c}	(0	0	0	0	0	0	0	0	1)	
							g_{3a}	(0	1	3	33	39	22	10	0)	/4	
							g_{3b}	(0	0	0	0	0	0	0	1	0)	/2
							g_{3c}	(0	0	0	0	0	0	0	0	1)	
							g_{4a}	(0	1	3	35	35	26	10	0)	/4	
							g_{4b}	(0	0	0	0	0	0	0	1	0)	/2
							g_{4c}	(0	0	0	0	0	0	0	0	1)	
							g_{5a}	(0	1	-2	18	20	7	2	-4)	/4	
							g_{5b}	(0	0	0	0	0	0	0	1	0)	/2
							g_{5c}	(0	0	0	0	0	0	0	0	0)	
							g_{6a}	(0	1	-2	18	20	7	2	-4)	/4	
							g_{6b}	(0	0	0	0	0	0	0	1	-2)	/2
							g_{6c}	(0	0	0	0	0	0	0	0	1)	
							g_{7a}	(0	0	1	-3	-16	-17	-11	-4)	/2	
							g_{7b}	(0	0	0	0	0	0	0	1	-2)	/2
							g_{7c}	(0	0	0	0	0	0	0	0	0)	
							g_{ma}	(0	1	2	16	20	13	6	0)	/4	
							g_{mb}	(0	0	0	0	0	0	0	0	0)	/2
							g_{mc}	(0	0	0	0	0	0	0	0	0)	/2

Table C.28: Degree 15, diameter class 5 (mod 7), $a = (2k - 3)/7$

Family A15:5a (transpose of A15:5b, conjugate of A15:2a, translate of A14:1a) Cyclic rank 2 (suffices a and b)								
LGM								
	$2a + 1$	-1	-1	-1	-1	0	-1	
	1	$2a + 1$	-1	-1	0	-1	-1	
	1	1	$2a + 1$	0	0	-1	0	
	1	1	0	$2a + 1$	0	-1	0	
	1	0	0	0	$2a + 1$	-1	-1	
	0	1	1	1	1	$2a + 1$	1	
	1	1	0	0	1	-1	$2a$	
Polynomial in $2a$								
Order, n	(1	6	29	80	125	112	53	10)
n_a	(0	1	5	24	56	69	43	10)
n_b	(0	0	0	0	0	0	1	1)
g_{1a}	(0	0	1	-1	4	14	19	7)
g_{1b}	(0	0	0	0	0	0	1	0)
g_{2a}	(0	0	1	1	-4	-10	-9	-3)
g_{2b}	(0	0	0	0	0	0	0	0)
g_{3a}	(0	0	1	5	10	10	5	1)
g_{3b}	(0	0	0	0	0	0	1	0)
g_{4a}	(0	0	1	5	10	10	5	1)
g_{4b}	(0	0	0	0	0	0	0	1)
g_{5a}	(0	0	1	3	4	2	-1	-1)
g_{5b}	(0	0	0	0	0	0	0	0)
g_{6a}	(0	0	1	9	36	58	47	13)
g_{6b}	(0	0	0	0	0	0	1	0)
g_{7a}	(0	0	1	7	18	24	17	5)
g_{7b}	(0	0	0	0	0	0	0	0)
g_{ma}	(0	1	5	24	56	69	43	10) /2
g_{mb}	(0	0	0	0	0	0	0	0)

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Table C.28: (cont.) Degree 15, diameter class 5 (mod 7), $a = (2k - 3)/7$

Family A15:5b (transpose of A15:5a, conjugate of A15:2b, translate of A14:1b) Cyclic rank 2 (suffices a and b)								
LGM								
$\begin{pmatrix} 2a+1 & 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & 2a+1 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 2a+1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2a+1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 2a+1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 2a+1 & -1 \\ -1 & -1 & 0 & 0 & -1 & 1 & 2a \end{pmatrix}$								
Polynomial in $2a$								
Order, n	(1	6	29	80	125	112	53	10)
n_a	(0	1	5	24	56	69	43	10)
n_b	(0	0	0	0	0	0	1	1)
g_{1a}	(0	0	1	6	21	31	24	7)
g_{1b}	(0	0	0	0	0	0	1	0)
g_{2a}	(0	0	1	5	14	30	27	7)
g_{2b}	(0	0	0	0	0	0	0	0)
g_{3a}	(0	0	1	2	9	11	2	-1)
g_{3b}	(0	0	0	0	0	0	1	0)
g_{4a}	(0	0	1	2	9	11	2	-1)
g_{4b}	(0	0	0	0	0	0	0	1)
g_{5a}	(0	0	1	5	28	58	45	11)
g_{5b}	(0	0	0	0	0	0	0	0)
g_{6a}	(0	0	0	2	1	-5	-9	-3)
g_{6b}	(0	0	0	0	0	0	0	1)
g_{7a}	(0	0	0	1	8	8	-2	-3)
g_{7b}	(0	0	0	0	0	0	0	0)
g_{ma}	(0	1	5	24	56	69	43	10)
g_{mb}	(0	0	0	0	0	0	0	0)

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Table C.28: (cont.) Degree 15, diameter class 5 (mod 7), $a = (2k - 3)/7$

Family A15:5c (self-transpose, conjugate of A15:2c, translate of A14:1c)								
Cyclic rank 3 (suffices a , b and c)								
LGM								
$\begin{pmatrix} 2a+2 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2a & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a+1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a+1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 2a+1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2a+1 \end{pmatrix}$								
Polynomial in $2a$								
Order, n	(1	6	30	84	113	70	16	0)
n_a	(0	1	5	25	59	54	16	0) /4
n_b	(0	0	0	0	0	0	2	2)
n_c	(0	0	0	0	0	0	0	2)
g_{1a}	(0	1	9	43	87	68	16	0) /8
g_{1b}	(0	0	0	0	0	0	1	1)
g_{1c}	(0	0	0	0	0	0	0	1)
g_{2a}	(0	1	9	29	49	40	8	0) /8
g_{2b}	(0	0	0	0	0	0	1	0)
g_{2c}	(0	0	0	0	0	0	0	1)
g_{3a}	(0	1	5	7	7	12	8	0) /8
g_{3b}	(0	0	0	0	0	0	1	0)
g_{3c}	(0	0	0	0	0	0	0	1)
g_{4a}	(0	1	5	7	7	12	8	0) /8
g_{4b}	(0	0	0	0	0	0	1	2)
g_{4c}	(0	0	0	0	0	0	0	1)
g_{5a}	(0	1	7	21	7	-36	-24	0) /8
g_{5b}	(0	0	0	0	0	0	1	2)
g_{5c}	(0	0	0	0	0	0	0	1)
g_{6a}	(0	1	3	9	45	102	96	32) /8
g_{6b}	(0	0	0	0	0	0	0	0)
g_{6c}	(0	0	0	0	0	0	0	1)
g_{7a}	(0	1	-1	-11	-55	-134	-120	-32) /8
g_{7b}	(0	0	0	0	0	0	1	1)
g_{7c}	(0	0	0	0	0	0	0	1)
g_{ma}	(0	1	5	25	59	54	16	0) /8
g_{mb}	(0	0	0	0	0	0	1	1)
g_{mc}	(0	0	0	0	0	0	0	0)

Table C.29: Degree 15, diameter class 6 (mod 7), $a = (2k + 2)/7$

LGM	Polynomial in $2a$
Family A15:6 (self-transpose?, conjugate of A15:1, translate of A14:2)	
Cyclic rank 3 (suffices a , b and c)	
$\begin{pmatrix} 2a-1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a-1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 2a-1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2a-1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2a & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2a & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2a \end{pmatrix}$	Order, n (1 -4 22 -48 41 -12 0 0) n_a (0 1 -3 19 -29 12 0 0) /16 n_b (0 0 0 0 0 0 4 -4) n_c (0 0 0 0 0 0 0 4) No formulae discovered yet for a generating set

Table C.30: Degree 19, diameter class 0 (mod 9), except $k \equiv 0$ (mod 27) (gcd 9), $a = 2k/9$

Family A19:0 (self-transpose, self-conjugate) Cyclic rank 7 (suffices a , b , c , d , e , f and g)										
LGM										
$\begin{pmatrix} 2a & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2a & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2a & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 2a & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 2a & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 2a & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2a & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2a & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 2a & 0 & 0 \end{pmatrix}$										
Polynomial in $2a$										
Order, n	(1	0	27	0	0	0	0	0	0	0)
n_a	(0	0	0	0	0	0	1	0	27	0)
n_b	(0	0	0	0	0	0	0	0	1	0)
n_c	(0	0	0	0	0	0	0	0	1	0)
n_d	(0	0	0	0	0	0	0	0	1	0)
n_e	(0	0	0	0	0	0	0	0	1	0)
n_f	(0	0	0	0	0	0	0	0	1	0)
n_g	(0	0	0	0	0	0	0	0	1	0)
g_{1a}	(0	0	0	0	0	0	0	1	6	9)
g_{1b}	(0	0	0	0	0	0	0	0	0	0)
g_{1c}	(0	0	0	0	0	0	0	0	0	0)
g_{1d}	(0	0	0	0	0	0	0	0	0	1)
g_{1e}	(0	0	0	0	0	0	0	0	0	1)
g_{1f}	(0	0	0	0	0	0	0	0	0	1)
g_{1g}	(0	0	0	0	0	0	0	0	0	1)
g_{2a}	(0	0	0	0	0	0	0	1	6	9)
g_{2b}	(0	0	0	0	0	0	0	0	0	0)
g_{2c}	(0	0	0	0	0	0	0	0	1	-1)
g_{2d}	(0	0	0	0	0	0	0	0	1	-1)
g_{2e}	(0	0	0	0	0	0	0	0	1	-1)
g_{2f}	(0	0	0	0	0	0	0	0	1	-1)
g_{2g}	(0	0	0	0	0	0	0	0	1	-1)

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Table C.30: (cont.) Degree 19, diameter class 0 (mod 9), except $k \equiv 0$
(mod 27) (gcd 9), $a = 2k/9$

g_{3a}	(0	0	0	0	0	0	0	1	6	9)
g_{3b}	(0	0	0	0	0	0	0	0	1	-1)
g_{3c}	(0	0	0	0	0	0	0	0	0	0)
g_{3d}	(0	0	0	0	0	0	0	0	1	-1)
g_{3e}	(0	0	0	0	0	0	0	0	1	-1)
g_{3f}	(0	0	0	0	0	0	0	0	1	-1)
g_{3g}	(0	0	0	0	0	0	0	0	1	-1)
g_{4a}	(0	0	0	0	0	0	0	1	0	-9)
g_{4b}	(0	0	0	0	0	0	0	0	1	-1)
g_{4c}	(0	0	0	0	0	0	0	0	1	-1)
g_{4d}	(0	0	0	0	0	0	0	0	0	0)
g_{4e}	(0	0	0	0	0	0	0	0	0	0)
g_{4f}	(0	0	0	0	0	0	0	0	1	-1)
g_{4g}	(0	0	0	0	0	0	0	0	1	-1)
g_{5a}	(0	0	0	0	0	0	0	1	0	-9)
g_{5b}	(0	0	0	0	0	0	0	0	0	1)
g_{5c}	(0	0	0	0	0	0	0	0	0	1)
g_{5d}	(0	0	0	0	0	0	0	0	0	0)
g_{5e}	(0	0	0	0	0	0	0	0	0	1)
g_{5f}	(0	0	0	0	0	0	0	0	0	1)
g_{5g}	(0	0	0	0	0	0	0	0	0	1)
g_{6a}	(0	0	0	0	0	0	0	1	0	-9)
g_{6b}	(0	0	0	0	0	0	0	0	0	1)
g_{6c}	(0	0	0	0	0	0	0	0	0	1)
g_{6d}	(0	0	0	0	0	0	0	0	0	1)
g_{6e}	(0	0	0	0	0	0	0	0	0	0)
g_{6f}	(0	0	0	0	0	0	0	0	0	1)
g_{6g}	(0	0	0	0	0	0	0	0	0	1)
g_{7a}	(0	0	0	0	0	0	0	1	-6	9)
g_{7b}	(0	0	0	0	0	0	0	0	0	1)
g_{7c}	(0	0	0	0	0	0	0	0	0	1)
g_{7d}	(0	0	0	0	0	0	0	0	0	1)
g_{7e}	(0	0	0	0	0	0	0	0	0	1)
g_{7f}	(0	0	0	0	0	0	0	0	0	0)
g_{7g}	(0	0	0	0	0	0	0	0	0	0)
g_{8a}	(0	0	0	0	0	0	0	1	-6	9)
g_{8b}	(0	0	0	0	0	0	0	0	1	-1)
g_{8c}	(0	0	0	0	0	0	0	0	1	-1)
g_{8d}	(0	0	0	0	0	0	0	0	1	-1)
g_{8e}	(0	0	0	0	0	0	0	0	1	-1)
g_{8f}	(0	0	0	0	0	0	0	0	0	0)
g_{8g}	(0	0	0	0	0	0	0	0	1	-1)
g_{9a}	(0	0	0	0	0	0	0	1	-6	9)
g_{9b}	(0	0	0	0	0	0	0	0	1	-1)
g_{9c}	(0	0	0	0	0	0	0	0	1	-1)
g_{9d}	(0	0	0	0	0	0	0	0	1	-1)
g_{9e}	(0	0	0	0	0	0	0	0	1	-1)
g_{9f}	(0	0	0	0	0	0	0	0	1	-1)
g_{9g}	(0	0	0	0	0	0	0	0	0	0)
g_{ma}	(0	0	0	0	0	0	1	0	27	0) /2
g_{mb}	(0	0	0	0	0	0	0	0	1	0) /2
g_{mc}	(0	0	0	0	0	0	0	0	1	0) /2
g_{md}	(0	0	0	0	0	0	0	0	1	0) /2
g_{me}	(0	0	0	0	0	0	0	0	1	0) /2
g_{mf}	(0	0	0	0	0	0	0	0	1	0) /2
g_{mq}	(0	0	0	0	0	0	0	0	1	0) /2

APPENDIX D

EXTREMAL AND LARGEST-KNOWN CIRCULANT GRAPHS

D.1 Circulant graphs up to degree 29

Of the extremal and largest-known circulant graphs presented in this appendix, the following have been discovered by the author:

Diameter 2: degrees 24 and 25

Diameter 4: degree 16

Diameter 5: degree 9 and above

Diameter 6 and above: degree 8 and above

Also independently by R. Fera-Purón, H. Pérez-Rosés and J. Ryan [13]:

Degree 8: diameter 3 to 5

Degree 9: diameter 4

And jointly with Grahame Erskine [1]:

Diameter 2: degrees 17 to 23

For verified extremal graphs the order is shown in bold text. For small diameter, the largest-known graph may have larger order than the member of the largest-known family.

Where a graph is a member of an identified largest-known family, the isomorphism class of the family is identified by a code beginning with ‘F’; otherwise the isomorphism class is specific to the graph and begins with ‘G’.

For each known isomorphism class just one generating set is defined: primitive if one exists, otherwise imprimitive. For odd degree, the involutory generator is omitted.

The automorphism group dihedral index (Aut group DI) is the order of the graph’s automorphism group expressed as a multiple of the order of the dihedral group on the same number of vertices. Where a graph is arc-transitive, this is indicated by ‘arc’ after the DI. For large graphs, where the index is conjectured on the basis of the structure of the generating set, this is indicated by a question mark.

Table D.1: Circulant graphs of degree 2 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Automorphism group DI
2	5	F2	1	0	2	1
3	7	F2	1	0	3	1
4	9	F2	1	0	4	1
5	11	F2	1	0	5	1
6	13	F2	1	0	6	1
7	15	F2	1	0	7	1
8	17	F2	1	0	8	1
9	19	F2	1	0	9	1
10	21	F2	1	0	10	1
11	23	F2	1	0	11	1
12	25	F2	1	0	12	1
13	27	F2	1	0	13	1
14	29	F2	1	0	14	1
15	31	F2	1	0	15	1
16	33	F2	1	0	16	1

Table D.2: Circulant graphs of degree 3 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Automorphism group DI
2	8	F3	1	0	2	1
3	12	F3	1	0	3	1
4	16	F3	1	0	4	1
5	20	F3	1	0	5	1
6	24	F3	1	0	6	1
7	28	F3	1	0	7	1
8	32	F3	1	0	8	1
9	36	F3	1	0	9	1
10	40	F3	1	0	10	1
11	44	F3	1	0	11	1
12	48	F3	1	0	12	1
13	52	F3	1	0	13	1
14	56	F3	1	0	14	1
15	60	F3	1	0	15	1
16	64	F3	1	0	16	1

Table D.3: Circulant graphs of degree 4 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Automorphism group DI
2	13	F4	1, 5	0	2	2 arc
3	25	F4	1, 7	0	3	2 arc
4	41	F4	1, 9	0	4	2 arc
5	61	F4	1, 11	0	5	2 arc
6	85	F4	1, 13	0	6	2 arc
7	113	F4	1, 15	0	7	2 arc
8	145	F4	1, 17	0	8	2 arc
9	181	F4	1, 19	0	9	2 arc
10	221	F4	1, 21	0	10	2 arc
11	265	F4	1, 23	0	11	2 arc
12	313	F4	1, 25	0	12	2 arc
13	365	F4	1, 27	0	13	2 arc
14	421	F4	1, 29	0	14	2 arc
15	481	F4	1, 31	0	15	2 arc
16	545	F4	1, 33	0	16	2 arc

Table D.4: Circulant graphs of degree 5 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Automorphism group DI
2	16	F5	1, 3	0	1	1
3	36	F5	1, 5	0	2	1
4	64	F5	1, 7	0	3	1
5	100	F5	1, 9	0	4	1
6	144	F5	1, 11	0	5	1
7	196	F5	1, 13	0	6	1
8	256	F5	1, 15	0	7	1
9	324	F5	1, 17	0	8	1
10	400	F5	1, 19	0	9	1
11	484	F5	1, 21	0	10	1
12	576	F5	1, 23	0	11	1
13	676	F5	1, 25	0	12	1
14	784	F5	1, 27	0	13	1
15	900	F5	1, 29	0	14	1
16	1024	F5	1, 31	0	15	1

Table D.5: Circulant graphs of degree 6 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Automorphism group DI
2	21	F6:2A	1, 3, 8	0	1	2
		F6:2B	1, 4, 6	0	1	1
		G6:2C	1, 2, 8	2	1	1
3	55	F6:0A	1, 5, 21	0	2	2
		F6:0B	1, 10, 16	0	2	1
4	117	F6:1	1, 16, 22	0	3	3 arc
5	203	F6:2A	1, 7, 57	0	3	2
		F6:2B	1, 22, 28	0	3	1
6	333	F6:0A	1, 9, 73	0	4	2
		F6:0B	1, 36, 46	0	4	1
7	515	F6:1	1, 46, 56	0	5	3 arc
8	737	F6:2A	1, 11, 133	0	5	2
		F6:2B	1, 56, 66	0	5	1
9	1027	F6:0A	1, 13, 157	0	6	2
		F6:0B	1, 78, 92	0	6	1
10	1393	F6:1	1, 92, 106	0	7	3 arc
11	1815	F6:2A	1, 15, 241	0	7	2
		F6:2B	1, 106, 120	0	7	1
12	2329	F6:0A	1, 17, 273	0	8	2
		F6:0B	1, 136, 154	0	8	1
13	2943	F6:1	1, 154, 172	0	9	3 arc
14	3629	F6:2A	1, 19, 381	0	9	2
		F6:2B	1, 172, 190	0	9	1
15	4431	F6:0A	1, 21, 421	0	10	2
		F6:0B	1, 210, 232	0	10	1
16	5357	F6:1	1, 232, 254	0	11	3 arc

Table D.6: Circulant graphs of degree 7 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Automorphism group DI
2	26	G7:2A	1, 2, 8	2	1	1
		G7:2B	1, 3, 8	0	1	1
3	76	F7:0	1, 27, 31	0	2	3 arc
4	160	F7:1A	1, 5, 31	0	2	2
		F7:1B	1, 45, 51	0	2	1
5	308	F7:2A	1, 7, 43	0	3	2
		F7:2B	1, 63, 69	0	3	1
6	536	F7:0	1, 231, 239	0	4	3 arc
7	828	F7:1A	1, 9, 91	0	4	2
		F7:1B	1, 225, 235	0	4	1
8	1232	F7:2A	1, 11, 111	0	5	2
		F7:2B	1, 275, 285	0	5	1
9	1764	F7:0	1, 803, 815	0	6	3 arc
10	2392	F7:1A	1, 13, 183	0	6	2
		F7:1B	1, 637, 651	0	6	1
11	3180	F7:2A	1, 15, 211	0	7	2
		F7:2B	1, 735, 749	0	7	1
12	4144	F7:0	1, 1935, 1951	0	8	3 arc
13	5236	F7:1A	1, 17, 307	0	8	2
		F7:1B	1, 1377, 1395	0	8	1
14	6536	F7:2A	1, 19, 343	0	9	2
		F7:2B	1, 1539, 1557	0	9	1
15	8060	F7:0	1, 3819, 3839	0	10	3 arc
16	9744	F7:1A	1, 21, 463	0	10	2
		F7:1B	1, 2541, 2563	0	10	1

Table D.7: Circulant graphs of degree 8 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Automorphism group DI
2	35	G8:2A	1, 6, 7, 10	2	1	2
		G8:2B	1, 7, 11, 16	0	1	3
3	104	F8:1	1, 16, 20, 27	0	2	2
4	248	F8:0	1, 61, 72, 76	0	2	2
5	528	F8:1	1, 89, 156, 162	0	3	2
6	984	F8:0	1, 163, 348, 354	0	3	2
7	1712	F8:1	1, 215, 608, 616	0	4	2
8	2768	F8:0	1, 345, 1072, 1080	0	4	2
9	4280	F8:1	1, 429, 1660, 1670	0	5	2
10	6320	F8:0	1, 631, 2580, 2590	0	5	2
11	9048	F8:1	1, 755, 3696, 3708	0	6	2
12	12552	F8:0	1, 1045, 5304, 5316	0	6	2
13	17024	F8:1	1, 1217, 7196, 7210	0	7	2
14	22568	F8:0	1, 1611, 9772, 9786	0	7	2
15	29408	F8:1	1, 1839, 12736, 12752	0	8	2
16	37664	F8:0	1, 2353, 16608, 16624	0	8	2

Table D.8: Circulant graphs of degree 9 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Automorphism group DI
2	42	G9:2A	1, 5, 14, 17	2	1	3
		G9:2B	2, 7, 8, 10	2	1	9
3	130	G9:3A	1, 8, 14, 47	2	2	2
		G9:3B	1, 8, 20, 35	2	2	1
		G9:3C	1, 26, 49, 61	2	2	3
		G9:3D	2, 8, 13, 32	2	2	3
4	320	G9:4	1, 15, 25, 83	2	3	1
5	700	F9:1a	1, 5, 197, 223	0	2	1
		F9:1b	1, 45, 225, 231	0	2	1
6	1416	F9:0	1, 7, 575, 611	0	3	1
7	2548	F9:1a	1, 7, 521, 571	0	3	1
		F9:1b	1, 581, 1021, 1029	0	3	1
8	4304	F9:0	1, 9, 1855, 1919	0	4	1
9	6804	F9:1a	1, 9, 1849, 1931	0	4	1
		F9:1b	1, 1305, 1855, 1863	0	4	1
10	10320	F9:0	1, 11, 4599, 4699	0	5	1
11	15004	F9:1a	1, 11, 3349, 3471	0	5	1
		F9:1b	1, 2299, 2309, 7029	0	5	1
12	21192	F9:0	1, 13, 9647, 9791	0	6	1
13	29068	F9:1a	1, 13, 7741, 7911	0	6	1
		F9:1b	1, 3875, 3887, 11479	0	6	1
14	39032	F9:0	1, 15, 18031, 18227	0	7	1
15	51300	F9:1a	1, 15, 11857, 12083	0	7	1
		F9:1b	1, 5835, 15075, 15089	0	7	1
16	66336	F9:0	1, 17, 30975, 31231	0	8	1

Table D.9: Circulant graphs of degree 10 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Automorphism group DI
2	51	G10:2	1, 2, 10, 16, 23	2	1	1
3	177	G10:3	1, 12, 19, 27, 87	2	2	1
4	457	F10:4	1, 20, 130, 147, 191	0	2	1
5	1099	F10:0	1, 53, 207, 272, 536	0	2	1
6	2380	F10:1	1, 555, 860, 951, 970	0	3	2
7	4551	F10:2	1, 739, 1178, 1295, 1301	0	3	2
8	8288	F10:3	1, 987, 2367, 2534, 3528	0	3	2
9	14099	F10:4	1, 1440, 3660, 3668, 6247	0	4	1
10	22805	F10:0	1, 218, 1970, 6819, 6827	0	4	1
11	35568	F10:1	1, 4347, 7470, 7903, 11808	0	5	2
12	53025	F10:2	1, 5251, 19281, 19291, 19806	0	5	2
13	77572	F10:3	1, 6347, 14103, 14740, 21098	0	5	2
14	110045	F10:4	1, 827, 9176, 9935, 18272	0	6	1
15	152671	F10:0	1, 973, 11663, 12716, 25364	0	6	1?
16	208052	F10:1	1, 17147, 30784, 32007, 47918	0	7	2?

Table D.10: Circulant graphs of degree 11 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Automorphism group DI
2	56	G11:2A	1, 2, 10, 15, 22	2	1	1
		G11:2B	1, 4, 6, 15, 24	2	1	1
		G11:2C	1, 6, 10, 15, 18	2	1	1
		G11:2D	1, 9, 14, 21, 25	2	1	3
		G11:2E	2, 6, 7, 18, 21	2	1	12
3	210	G11:3A	1, 49, 59, 84, 89	2	2	3
		G11:3B	2, 32, 63, 92, 98	2	2	3
4	576	G11:4	1, 9, 75, 155, 179	2	3	1
5	1428	F11:0	1, 169, 285, 289, 387	0	2	2
6	3200	F11:1a	1, 101, 925, 1031, 1429	0	2	1
		F11:1b	1, 265, 851, 1111, 1321	0	2	1
7	6652	F11:2	1, 107, 647, 2235, 2769	0	3	1
8	12416	F11:3	1, 145, 863, 4163, 5177	0	3	1
9	21572	F11:4	1, 189, 1517, 8113, 9435	0	4	1
10	35880	F11:0	1, 2209, 5127, 5135, 12537	0	4	2
11	56700	F11:1a	1, 1053, 1061, 10603, 17965	0	4	1
		F11:1b	1, 4113, 4121, 13301, 23723	0	4	1
12	87248	F11:2	1, 479, 4799, 34947, 39257	0	5	1
13	128852	F11:3	1, 581, 5799, 51599, 57989	0	5	1
14	184424	F11:4	1, 693, 8325, 76901, 84523	0	6	1?
15	259260	F11:0	1, 10729, 39875, 39887, 90637	0	6	2?
16	355576	F11:1a	1, 22307, 131327, 136371, 153621	0	6	1?
		F11:1b	1, 8579, 75569, 75583, 111513	0	6	1?

Table D.11: Circulant graphs of degree 12 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	67	G12:2	1, 2, 3, 13, 21, 30	2	1	1
3	275	G12:3	1, 16, 19, 29, 86, 110	2	2	5
4	819	G12:4	7, 26, 119, 143, 377, 385	0	2	9
5	2120	G12:5	1, 488, 529, 704, 868, 940	0	2	2
6	5044	F12:0	1, 26, 99, 266, 1034, 1163	0	2	1
7	10777	G12:7	1, 703, 1533, 1981, 2241, 2410	0	3	6 arc
8	21384	G12:8	130, 333, 489, 1046, 2648, 3831	0	3	1
9	39996	F12:0	549, 699, 1456, 1688, 5235, 6898	0	3	1
10	69965	F12:1a	1, 4935, 14224, 19166, 19991, 23842	0	4	2
		F12:1b	1, 7728, 19991, 25270, 28126, 34104	0	4	2
11	117712	F12:2a	1008, 1296, 8071, 15520, 22785, 39928	0	4	4
		F12:2b	679, 2184, 2808, 3584, 14008, 15393	0	4	4
12	190392	F12:0	1, 871, 23908, 39652, 45740, 70527	0	4	1?
13	295965	F12:1a	1, 8613, 20367, 65771, 83682, 92304	0	5	2?
		F12:1b	1, 65771, 68022, 75330, 93348, 134604	0	5	2?
14	448920	F12:2a	1260, 1540, 28719, 48340, 73611, 116190	0	5	4?
		F12:2b	2259, 5310, 6490, 9000, 43390, 47151	0	5	4?
15	662680	F12:0	2315, 5345, 8426, 11694, 58145, 69536	0	5	1?
16	952985	F12:1a	1, 47498, 155243, 173271, 227766, 382998	0	6	2?
		F12:1b	1, 118635, 134387, 173271, 293337, 411961	0	6	2?

Table D.12: Circulant graphs of degree 13 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Aut group DI
2	80	G13:2A	1, 3, 9, 20, 25, 33	2	1	1
		G13:2B	1, 5, 7, 20, 23, 31	2	1	2
		G13:2C	1, 5, 13, 20, 29, 37	2	1	1
3	312	G13:3A	1, 14, 74, 77, 130, 138	2	2	1
		G13:3B	2, 9, 54, 69, 134, 146	2	2	1
4	970	G13:4	1, 23, 40, 76, 172, 395	2	3	1
5	2676	G13:5	1, 231, 333, 753, 893, 927	0	2	2
6	6256	G13:6	1, 157, 161, 197, 327, 1115	0	2	1
7	14740	F13:1a	1, 1095, 2949, 4385, 4605, 5755	0	2	2
		F13:1b	1, 605, 1865, 2465, 2949, 6905	0	2	2
8	30760	F13:2a	1, 4135, 6151, 9565, 12105, 14525	0	3	2
		F13:2b	1, 2005, 6151, 11925, 12575, 13935	0	3	2
9	57396	F13:0a	1, 665, 797, 3319, 19243, 24029	0	3	1
		F13:0b	1, 1847, 19867, 21709, 24599, 28391	0	3	1
10	106120	F13:1a	1, 15161, 20153, 29967, 38731, 50113	0	3	2
		F13:1b	1, 5495, 15161, 32977, 33761, 39389	0	3	2
11	182980	F13:2a	1, 19663, 26139, 51821, 71477, 89033	0	4	2?
		F13:2b	1, 16709, 20783, 26139, 39879, 56595	0	4	2?
12	295840	F13:0a	1, 21999, 97841, 111809, 111817, 140767	0	4	1?
		F13:0b	1, 12737, 13729, 58639, 100903, 123553	0	4	1?
13	476100	F13:1a	1, 52901, 66033, 105075, 171099, 229797	0	4	2?
		F13:1b	2737, 42111, 51093, 55637, 56205, 68571	0	4	2?
14	732744	F13:2a	1, 65223, 81415, 161937, 227151, 299619	0	5	2?
		F13:2b	1, 81415, 155241, 168345, 217953, 359541	0	5	2?
15	1081860	F13:0a	1, 4869, 5409, 43811, 433231, 487329	0	5	1?
		F13:0b	1, 14381, 143819, 159639, 196701, 303449	0	5	1?
16	1593064	F13:1a	1, 144825, 173657, 288343, 461989, 619861	0	5	2?
		F13:1b	1, 20427, 59433, 144825, 594319, 614757	0	5	2?

Table D.13: Circulant graphs of degree 14 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	90	G14:2A	1, 4, 10, 17, 26, 29, 41	0	1	1
		G14:2B	1, 5, 8, 19, 25, 28, 40	2	1	2
		G14:2C	1, 5, 11, 14, 32, 34, 41	0	1	1
		G14:2D	3, 5, 9, 12, 25, 35, 36	2	1	18
3	381	G14:3	1, 11, 103, 120, 155, 161, 187	2	2	1
4	1229	G14:4	1, 8, 105, 148, 160, 379, 502	2	2	1
5	3695	F14:5	1, 38, 365, 1038, 1060, 1073, 1188	0	2	1
6	9800	F14:6a	441, 1772, 2088, 2508, 2891, 5032, 8788	0	2	2
		F14:6b	1, 1472, 1756, 2451, 2928, 3216, 3756	2	2	2
7	23304	F14:0a	1, 5504, 5827, 6192, 6364, 7056, 10732	0	3	2
		F14:0b	1, 1280, 1824, 3004, 5827, 9124, 9900	0	3	2
8	49757	F14:1	1, 845, 4192, 8267, 8468, 9266, 12491	0	3	1
9	103380	F14:2a	725, 4848, 6870, 15828, 16505, 30000, 45330	0	3	4
		F14:2b	12, 1110, 4595, 12635, 15150, 20688, 22020	0	3	4
10	196689	F14:3	1, 5165, 24410, 35629, 54868, 72479, 77119	0	4	7? arc
11	350700	F14:4a	1603, 4830, 10812, 39288, 43428, 60053, 62286	0	4	4?
		F14:4b	984, 5754, 14903, 43547, 49116, 88410, 129696	0	4	4?
12	593989	F14:5	1, 1764, 38857, 134389, 171474, 175261, 273764	0	4	1?
13	996240	F14:6a	1, 263656, 334656, 350120, 373591, 449488, 460296	0	4	2?
		F14:6b	1, 67624, 130368, 371584, 373591, 426672, 487912	0	4	2?
14	1603216	F14:0a	1, 115856, 392096, 415688, 490840, 601207, 679584	0	5	2?
		F14:0b	1, 313192, 321472, 412872, 601207, 677392, 777080	0	5	2?
15	2486227	F14:1	1, 99775, 175188, 332678, 477082, 722778, 1199869	0	5	1?
16	3843540	F14:2a	7713, 61880, 69030, 365180, 392067, 698400, 924570	0	5	4?
		F14:2b	12510, 16420, 56997, 327357, 347310, 443480, 457380	0	5	4?

Table D.14: Circulant graphs of degree 15 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maxi- mal levels	Aut group DI
2	96	G15:2A	1, 2, 3, 14, 21, 31, 39	2	1	1
		G15:2B	1, 3, 5, 11, 24, 31, 39	2	1	1
		G15:2C	1, 3, 5, 13, 24, 33, 41	2	1	1
		G15:2D	1, 3, 5, 15, 24, 31, 37	2	1	2
		G15:2E	1, 3, 7, 15, 24, 35, 43	2	1	1
		G15:2F	1, 3, 11, 15, 17, 24, 43	2	1	1
		G15:2G	1, 3, 11, 15, 24, 29, 31	2	1	1
		G15:2H	1, 3, 13, 19, 24, 33, 41	2	1	1
		G15:2I	1, 4, 6, 9, 20, 31, 42	2	1	1
		G15:2J	1, 5, 9, 21, 24, 31, 37	2	1	2
		G15:2K	1, 5, 11, 24, 31, 39, 45	2	1	1
		G15:2L	1, 5, 13, 17, 24, 27, 33	2	1	1
		G15:2M	1, 5, 24, 33, 35, 41, 45	2	1	1
		G15:2N	1, 7, 11, 24, 27, 39, 43	2	1	1
3	448	G15:3	1, 10, 127, 150, 176, 189, 217	2	2	1
4	1420	G15:4	1, 20, 111, 196, 264, 340, 343	4	2	1
5	4292	F15:5	1, 49, 631, 721, 871, 1307, 1941	2	2	1
6	12232	F15:6	1, 25, 965, 987, 3039, 5357, 5411	0	2	1
7	32092	F15:0	1, 1949, 9409, 11745, 12547, 13609, 15949	0	3	7
8	68944	F15:1	1, 605, 3889, 13667, 25329, 26799, 29827	0	3	1
9	142516	F15:2	1, 4693, 20067, 38833, 43603, 48301, 52039	0	3	1
10	276928	F15:3a	1, 923, 18307, 76639, 78767, 91541, 122439	0	3	1?
		F15:3b	1, 2023, 41295, 66071, 79691, 119029, 129137	0	3	1?
11	514580	F15:4	1, 15777, 38497, 44443, 71941, 200663, 223377	0	4	1?
12	908480	F15:5	1, 12585, 137279, 177161, 265263, 311571, 371535	0	4	1?
13	1550228	F15:6	1, 17363, 84027, 644541, 672223, 699419, 744457	0	4	1?
14	2566712	F15:0	1, 193103, 241127, 375311, 423327, 483801, 1069905	0	5	7?
15	4013468	F15:1	1, 31267, 536859, 639891, 895355, 1372563, 1464147	0	5	1?
16	6155056	F15:2	1, 142993, 1147423, 1178161, 2291357, 2465097, 2993591	0	5	1?

Table D.15: Circulant graphs of degree 16 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Aut group DI
2	112	G16:2	1, 4, 10, 17, 29, 36, 45, 52	0	1	1
3	518	G16:3	1, 8, 36, 46, 75, 133, 183, 247	2	2	1
4	1788	G16:4	1, 16, 249, 288, 465, 585, 590, 799	2	2	1
5	5847	F16:1a	1, 720, 1053, 1188, 1311, 1581, 1948, 2742	2	2	2
		F16:1b	1, 1041, 1083, 1560, 1948, 2214, 2628, 2835	2	2	2
6	17733	F16:2a	1, 1695, 2652, 2868, 5418, 5912, 6756, 8232	2	2	2
		F16:2b	1, 1695, 1734, 3132, 3288, 5912, 8172, 8466	2	2	2
7	45900	G16:7	1, 4902, 6266, 12499, 14334, 18802, 19758, 21927	2	3	1
8	107748	G16:4	1, 4382, 16334, 19322, 20865, 26082, 43458, 47803	0	3	1
9	232245	F16:1a	1, 22915, 28645, 47680, 54755, 56760, 58630, 92899	0	3	2?
		F16:1b	2850, 5655, 17645, 17922, 28527, 57715, 93745, 113650	0	3	2?
10	479255	F16:2a	1, 31900, 46575, 63990, 151740, 152040, 166220, 191701	0	3	2?
		F16:2b	1, 19365, 82565, 119090, 139035, 191701, 194930, 232055	0	3	2?
11	924420	F16:3a	12810, 27936, 33955, 56580, 120115, 122910, 275430, 397704	0	4	4?
		F16:3b	5065, 6060, 38652, 44730, 100110, 146232, 149005, 187290	0	4	4?
12	1702428	F16:0a	3654, 38640, 60888, 75299, 206262, 208439, 304092, 335370	0	4	4?
		F16:0b	15701, 18480, 61596, 75054, 235410, 268037, 424812, 636258	0	4	4?
13	2982623	F16:1a	1, 22414, 217924, 845313, 852179, 862246, 1002218, 1237068	0	4	2?
		F16:1b	1, 707126, 780465, 829864, 852179, 1003520, 1137402, 1238419	0	4	2?
14	5109237	F16:2a	8676, 48237, 49728, 126623, 535451, 738567, 1174292, 1570625	0	4	2?
		F16:2b	1, 82208, 536753, 846349, 1459781, 1987895, 2145444, 2379706	0	4	2?
15	8476048	F16:3a	27328, 187512, 187985, 264152, 871521, 920528, 1921920, 2609240	0	5	4?
		F16:3b	1239, 37464, 222880, 224296, 768432, 986568, 1060745, 1260448	0	5	4?
16	13588848	F16:0a	22176, 140184, 320024, 335079, 1321056, 1363527, 1829896, 1969776	0	5	4?
		F16:0b	15759, 87912, 324680, 338688, 1505808, 1682847, 2695064, 3751488	0	5	4?

Table D.16: Circulant graphs of degree 17 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Aut group DI
2	130	G17:2A	1, 7, 26, 37, 47, 49, 52, 61	2	1	6
		G17:2B	2, 8, 13, 14, 32, 36, 39, 56	0	1	30
3	570	G17:3	1, 26, 63, 72, 105, 218, 234, 266	2	2	1
4	1954	G17:4	1, 35, 80, 122, 144, 437, 634, 694	2	2	1
5	6468	F17:1a	1, 489, 981, 2007, 2155, 2199, 2325, 2973	2	2	2
		F17:1b	1, 459, 1185, 1455, 1659, 1851, 2155, 2403	2	2	2
6	20360	F17:2a	1, 85, 259, 2541, 3941, 4719, 6199, 6227	2	2	1
		F17:2b	1, 443, 695, 3055, 5137, 7081, 9353, 9609	2	2	1
7	57684	F17:3a	1, 1347, 5385, 5721, 7629, 8031, 10707, 19229	2	2	2
		F17:3b	1, 1203, 14121, 16125, 16317, 16749, 16893, 19229	2	2	2
8	136512	F17:0a	1, 2569, 3427, 24561, 28077, 42005, 50073, 55291	2	3	1
		F17:0b	1, 1993, 8519, 17475, 29125, 44777, 45323, 57109	2	3	1
9	321780	F17:1a	6875, 15835, 25257, 28965, 34445, 40165, 89613, 103405	2	3	2?
		F17:1b	1, 28645, 31315, 91115, 113895, 114585, 121745, 128711	0	3	2?
10	659464	F17:2a	1, 22977, 31087, 105401, 128373, 187579, 214641, 247349	2	3	1?
		F17:2b	1, 15657, 18675, 233371, 251261, 263339, 326751, 328979	2	3	1?
11	1350820	F17:3a	1, 9595, 57575, 59175, 199155, 323455, 405865, 540329	0	4	2?
		F17:3b	1, 98935, 241625, 319325, 536205, 540329, 589425, 643445	0	4	2?
12	2479104	F17:0a	1, 13015, 224021, 348731, 976025, 991643, 1087727, 1181429	0	4	1?
		F17:0b	1, 27395, 330677, 354157, 660977, 665585, 964693, 1156711	0	4	1?
13	4557364	F17:1a	1, 397397, 533267, 567105, 1585353, 1945391, 1953155, 2180745	0	4	2?
		F17:1b	1, 143185, 934451, 1455475, 1698053, 1953155, 2075885, 2122939	0	4	2?
14	7729000	F17:2a	1, 44599, 172671, 312201, 1128809, 1164099, 2057343, 3394105	0	4	1?
		F17:2b	1, 178557, 301355, 311615, 1061091, 2181313, 2298293, 2421099	0	4	1?
15	13275108	F17:3a	1, 738143, 2567453, 2672901, 3905909, 4866981, 5586539, 5689333	0	5	2?
		F17:3b	513919, 692755, 839547, 1056897, 1324113, 1535723, 1765477, 1944271	0	5	2?
16	21252864	F17:0a	1, 58311, 466481, 2724247, 5605767, 6671679, 7602345, 9860367	0	5	1?
		F17:0b	1, 155775, 1565119, 1741287, 4355481, 4667849, 7166801, 9179097	0	5	1?

Table D.17: Circulant graphs of degree 18 for diameter $k \leq 19$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Aut group DI
2	138	G18:2A	1, 9, 12, 15, 22, 42, 47, 51, 68	2	1	8
		G18:2B	1, 9, 22, 42, 47, 51, 54, 57, 68	2	1	8
		G18:2C	1, 15, 21, 24, 28, 33, 36, 47, 64	2	1	2
3	655	G18:3	1, 5, 48, 75, 125, 142, 160, 189, 300	2	1	1
4	2645	F18:4	1, 16, 234, 275, 328, 585, 902, 1008, 1055	2	2	1
5	8425	G18:5A	1, 39, 626, 697, 1282, 1954, 2169, 3770, 4106	2	2	1
		G18:5B	1, 28, 563, 867, 1123, 2270, 2722, 3882, 4169	2	2	1
6	27273	F18:6a	1, 1461, 2193, 4944, 6690, 8055, 9092, 12090, 13383	4	2	2
		F18:6b	1, 3351, 3750, 6699, 8010, 9092, 9258, 9756, 12198	2	2	2
7	80940	F18:7a	483, 2250, 2305, 7548, 12369, 28599, 29285, 29874, 45891	2	2	2
		F18:7b	1, 1389, 5934, 6663, 19194, 20709, 26981, 35766, 38805	2	2	2
8	208872	F18:8a	6879, 8912, 10468, 13280, 16464, 16812, 16928, 42068, 45339	2	3	2?
		F18:8b	1, 6288, 20084, 44940, 46344, 52219, 85156, 85404, 100608	4	3	2?
9	492776	F18:0a	1, 3248, 13008, 68276, 112716, 123195, 136112, 216420, 230060	0	3	2?
		F18:0b	1, 11936, 39880, 48020, 115524, 123195, 132400, 148084, 184408	2	3	2?
10	1078280	F18:1a	18985, 19500, 80740, 96160, 96340, 107248, 108408, 288555, 350340	2	3	2?
		F18:1b	16240, 25475, 36400, 36620, 57428, 88160, 158228, 244095, 433840	2	3	2?
11	2202955	F18:2a	1, 243915, 343340, 560850, 774970, 809705, 829590, 834535, 881181	0	3	2?
		F18:2b	1, 116575, 326310, 466295, 504675, 565515, 828570, 881181, 930200	2	3	2?
12	4388640	F18:3a	86850, 247675, 289128, 291390, 309990, 414300, 924540, 979115, 2922312	2	4	1?
		F18:3b	19020, 64230, 78305, 262800, 302514, 396480, 575214, 653135, 1398450	2	4	4?
13	8068383	F18:4	1, 19229, 37961, 43597, 2658766, 2692582, 3462710, 3544262, 3709706	0	4	1?
14	14718984	F18:5a	96852, 127344, 371154, 703194, 707399, 882252, 1745765, 2805906, 4158294	0	4	4?
		F18:5b	2784, 69930, 81270, 439068, 455994, 507108, 535241, 1917923, 2099928	0	4	4?
15	25609955	F18:6a	1, 409661, 477939, 2177616, 5934754, 6243342, 7317129, 10384150, 12544266	0	4	2?
		F18:6b	1, 1523473, 2600318, 4242028, 7317129, 8746850, 10456264, 11517667, 12187777	0	4	2?
16	43068508	F18:7a	1, 2585681, 7655844, 11414949, 11589914, 12305287, 14902181, 18482926, 20685441	0	4	2?
		F18:7b	1, 1150282, 1314607, 1442441, 8136562, 9159003, 10947433, 12305287, 12657414	0	4	2?

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Table D.17: (cont.) Circulant graphs of degree 18 for diameter $k \leq 19$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Aut group DI
17	70861072	F18:8a	1, 633504, 3514568, 4355944, 6960384, 15235032, 23682016, 26572903, 30413024	0	5	2?
		F18:8b	1, 4047176, 8309856, 8999640, 24480496, 24602840, 26572903, 29361680, 30173816	0	5	2?
18	113542416	F18:0a	1, 28237632, 32426832, 34015968, 36424392, 42578407, 49137480, 50918248, 54158728	0	5	2?
		F18:0b	472896, 635968, 1030784, 2335360, 2568168, 2683749, 2706776, 11509053, 12750824	0	5	2?
19	177875280	F18:1a	249480, 3511287, 3742344, 3964104, 6419952, 6888128, 12875792, 25745697, 36229032	0	5	2?
		F18:1b	342432, 1670625, 1827432, 1896192, 2890040, 4684896, 16873880, 20563785, 40561056	0	5	2?

Table D.18: Circulant graphs of degree 19 for diameter $k \leq 17$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Aut group DI
2	156	G19:2	1, 15, 21, 23, 26, 33, 52, 61, 65	2	1	3
3	722	G19:3	1, 10, 48, 101, 132, 135, 167, 171, 244	2	1	1
4	2696	G19:4	1, 5, 100, 322, 404, 570, 602, 1159, 1218	4	2	1
5	9652	G19:5A	1, 235, 247, 273, 3131, 3249, 3363, 3473, 4065	2	2	2
		G19:5B	1, 475, 817, 1387, 1637, 2589, 3605, 4065, 4177	2	2	2
6	31440	F19:6a	1, 2055, 3219, 4503, 5727, 5997, 7071, 10481, 15501	4	2	2
		F19:6b	1, 1587, 1995, 3369, 5559, 7569, 8175, 9609, 10481	2	2	2
7	99420	G19:7A	1, 243, 10173, 13725, 18393, 22785, 31245, 33139, 35961	2	2	2
		G19:7B	1, 6717, 10287, 22995, 24285, 24819, 33139, 44841, 48039	2	2	2
8	258040	G19:8A	1, 6013, 61777, 77995, 92643, 98335, 110521, 112535, 122737	2	2	1?
		G19:8B	1, 1591, 9215, 10803, 50135, 82225, 94361, 109103, 119407	2	2	1?
9	652004	F19:0	1, 1225, 7433, 116093, 123199, 159059, 190165, 203399, 318263	2	3	1?
10	1416256	F19:1a	1, 4967, 178527, 184735, 225221, 287225, 314983, 630623, 697185	2	3	1?
		F19:1b	1, 8891, 127041, 158801, 172255, 226071, 436827, 627333, 670769	2	3	1?
11	3101860	F19:2a	1, 432795, 620371, 667265, 770075, 856635, 876395, 945355, 962595	4	3	2?
		F19:2b	1, 74855, 299425, 307615, 620371, 873245, 1091555, 1145255, 1360055	2	3	2?
12	6100520	F19:3a	1, 144845, 199185, 533245, 711855, 1037895, 1589135, 2440209, 2509015	2	3	2?

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Table D.18: (cont.) Circulant graphs of degree 19 for diameter $k \leq 17$

k	Order	Iso class	Generating set (plus the involution)	Odd girth defect	Maximal levels	Aut group DI
		F19:3b	1, 11375, 132565, 751375, 808255, 2316445, 2440209, 2686855, 2876295	2	3	2?
13	11797684	F19:4	1, 42157, 210791, 1179953, 2435871, 3911403, 4472967, 4549331, 4557761	0	4	1?
14	21659528	F19:5	1, 7207, 768141, 1317277, 2385367, 5376981, 7896461, 8707845, 10559115	0	4	1?
15	38328220	F19:6a	1, 2416211, 5806843, 12177585, 13814381, 15627619, 16426381, 17601101, 17768919	0	4	2?
		F19:6b	1, 280721, 2634891, 9297617, 12009417, 14208005, 15026403, 16426381, 18101909	0	4	2?
16	66601304	F19:7a	1, 9514471, 13753159, 19092899, 19565399, 19936119, 23320591, 28684341, 30118501	0	4	2?
		F19:7b	1, 3216227, 9514471, 13095649, 18301073, 19297355, 24338601, 29530235, 30163343	0	4	2?
17	109535540	F19:8a	1, 589431, 1921675, 13815509, 14974299, 15962839, 31211663, 43393371, 46760881	0	4	1?
		F19:8b	1, 331711, 15268837, 23100849, 23301385, 24847989, 27342421, 43416841, 52186807	0	4	1?

Table D.19: Circulant graphs of degree 20 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Aut group DI
2	171	G20:2	1, 11, 31, 36, 37, 50, 54, 57, 65, 81	2	1	6
3	815	G20:3	1, 40, 65, 96, 109, 144, 155, 182, 202, 379	2	2	1
4	3175	G20:4	1, 18, 172, 274, 449, 644, 693, 784, 911, 1121	4	2	1
5	12396	F20:0a	1, 28, 1586, 1626, 2082, 3376, 4552, 4800, 5670, 6197	2	1	2
		F20:0b	358, 476, 702, 962, 1308, 1336, 1392, 2211, 3494, 3987	2	1	2
6	42252	G20:6a	1, 3024, 3174, 9520, 10214, 10530, 12618, 14574, 19092, 21125	2	1	2
		G20:6b	1, 1746, 4056, 10198, 11656, 15678, 17862, 18698, 19644, 21125	2	1	2
7	132720	G20:7A	825, 6195, 6663, 7092, 7918, 11832, 15285, 16554, 27009, 36322	2	2	2
		G20:7B	6897, 7722, 11433, 11538, 16287, 16306, 21168, 25545, 27934, 33273	4	2	2
8	371400	G20:8A	1, 17652, 61064, 74640, 90924, 92849, 102240, 109920, 155588, 175664	4	2	2?
		G20:8B	1, 21216, 30024, 40236, 67436, 76068, 92849, 143344, 146280, 174936	4	2	2?
9	930184	F20:4a	1, 66608, 117956, 179476, 232547, 273784, 303520, 313432, 445912, 461708	4	3	2?
		F20:4b	1, 21648, 29652, 31536, 210428, 232547, 329040, 342668, 349436, 408956	2	3	2?
10	2232648	F20:0a	1, 118028, 133224, 189980, 477360, 558161, 724588, 740212, 783736, 1082712	2	3	2?

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Table D.19: (cont.) Circulant graphs of degree 20 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Odd girth defect	Maximal levels	Aut group DI
11	4947880	F20:0b	1, 545916, 558161, 567476, 583176, 596988, 667932, 674272, 1098688, 1100272	2	3	2?
		F20:1a	2260, 44680, 97255, 123560, 165620, 223920, 243888, 377720, 745688, 1139715	2	3	4?
		F20:1b	105276, 137740, 199820, 200720, 215980, 423715, 691260, 813255, 1873876, 2253920	2	3	4?
12	10238745	F20:2a	1, 267020, 448155, 552080, 1032955, 1086360, 2190785, 2492930, 3119555, 4095499	4	3	2?
		F20:2b	8875, 13430, 101335, 448870, 460262, 544300, 1728800, 8651258, 10114635, 10149065	4	3	2?
13	20452920	F20:3a	63985, 602760, 695070, 1063380, 1127520, 3472805, 3859260, 5700810, 8416284	0	4	4?
		F20:3b	210, 4230, 151380, 152442, 756930, 783005, 909330, 938760, 2625815, 3938142	2	4	4?
14	38155632	F20:4a	1, 138846, 2971386, 6359273, 7552620, 10529382, 10964544, 13658208, 16023144, 16167372	0	4	2?
		F20:4b	1, 2278728, 5457870, 5816946, 5829414, 6359273, 12307044, 15561006, 17509818, 17584632	2	4	2?
15	70612644	F20:0a	1, 11768773, 21044952, 22808862, 24062964, 25004334, 25038822, 27857124, 33070614, 34646424	0	4	2?
		F20:0b	1, 6398412, 11768773, 14333250, 16580220, 17951334, 27036228, 27410724, 30696954, 35262918	0	4	2?
16	126967008	F20:1a	22386, 338772, 413371, 2263422, 2636928, 3506886, 3852618, 4851630, 14631258, 20747797	0	4	4?
		F20:1b	1018500, 1667088, 2158800, 2284506, 3687516, 5192173, 15161496, 15968995, 34609200, 46264638	0	4	4?

Table D.20: Circulant graphs of degrees 21 to 29 for some diameters k

Degree	k	Order	Iso class	Generating set (plus the involution for odd degree)	Odd girth defect	Maximal levels	Aut group DI
21	2	192	G21:2	1, 3, 15, 23, 32, 51, 57, 64, 85, 91	2	1	1
	5	8532	F21:0	1, 3, 5, 19, 33, 123, 213, 795, 1377, 2385	2	1	1
	10	2069424	F21:0	1, 5, 182073, 413871, 413887, 579429, 652419, 662163, 678969, 850203	0	2	1?
	15	83328852	F21:0	1, 7, 3263105, 5343719, 14234911, 15305275, 22350731, 27209599, 35712361, 35712397	0	3	1?
22	2	210	G22:2	2, 7, 12, 18, 32, 35, 63, 70, 78, 91, 92	2	1	3
	5	13749	F22:5	1, 3, 310, 2218, 2389, 3001, 3356, 3358, 3677, 3685, 6483	0	1	1

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Table D.20: (cont.) Circulant graphs of degree 21 to 29 for some diameters k

Degree	k	Order	Iso class	Generating set (plus the involution for odd degree)	Odd girth defect	Maximal levels	Aut group DI
16	285487743	F22:5		1, 558104, 21691418, 26156396, 59640662, 77911027, 77911033, 126700978, 129107465, 129108149, 129589720	0	3	1?
23	2	216	G23:2A	1, 3, 5, 17, 27, 36, 43, 57, 72, 83, 95	2	1	1
			G23:2B	1, 3, 5, 17, 27, 36, 49, 61, 72, 87, 101	2	1	1
			G23:2C	1, 3, 5, 19, 36, 45, 49, 51, 61, 72, 79	2	1	1
			G23:2D	1, 3, 5, 21, 27, 36, 49, 61, 72, 89, 101	2	1	1
			G23:2E	1, 3, 7, 17, 36, 49, 61, 72, 77, 87, 99	2	1	1
			G23:2F	1, 3, 7, 19, 36, 47, 49, 57, 72, 77, 81	2	1	1
			G23:2G	1, 3, 7, 25, 36, 45, 51, 55, 67, 72, 85	2	1	1
			G23:2H	1, 3, 9, 23, 31, 36, 47, 51, 72, 89, 101	2	1	1
			G23:2I	1, 3, 11, 19, 36, 45, 51, 72, 79, 85, 103	2	1	1
			G23:2J	1, 3, 17, 29, 36, 47, 57, 59, 67, 72, 81	2	1	1
			G23:2K	1, 5, 7, 23, 27, 36, 72, 75, 83, 87, 89	2	1	1
			G23:2L	1, 5, 13, 19, 25, 33, 36, 65, 72, 81, 87	2	1	1
			G23:2M	1, 5, 19, 25, 27, 36, 39, 72, 79, 87, 95	2	1	1
			G23:2N	1, 5, 21, 23, 29, 36, 53, 61, 63, 72, 75	2	1	1
			G23:2O	1, 5, 23, 36, 39, 45, 51, 53, 65, 72, 83	2	1	1
			G23:2P	1, 7, 13, 17, 36, 57, 61, 72, 75, 81, 103	2	1	1
			G23:2Q	1, 15, 36, 55, 61, 65, 69, 72, 81, 85, 103	2	1	1
11	10556484	F23:0		1, 20697, 51743, 134531, 610565, 633359, 693349, 1883443, 2183585, 4717021, 5267403	0	2	1?
24	2	231	G24:2A	1, 2, 9, 15, 35, 47, 57, 62, 76, 78, 90, 101	2	1	1
			G24:2B	1, 7, 12, 28, 35, 42, 45, 49, 67, 100, 105, 111	2	1	3
			G24:2C	1, 12, 28, 35, 42, 45, 49, 67, 100, 105, 111, 112	2	1	3
			G24:2D	1, 28, 35, 42, 43, 49, 67, 89, 100, 105, 109, 112	2	1	6
			G24:2E	1, 28, 35, 43, 49, 63, 67, 84, 89, 100, 109, 112	2	1	6
			G24:2F	1, 28, 35, 43, 49, 67, 84, 89, 100, 105, 109, 112	2	1	6
25	2	242	G25:2A	1, 2, 3, 4, 5, 23, 34, 51, 63, 79, 92, 106	2	1	1
			G25:2B	1, 2, 3, 4, 5, 23, 34, 51, 63, 79, 92, 107	2	1	1
			G25:2C	1, 2, 3, 4, 21, 30, 46, 56, 70, 82, 97, 108	2	1	1
			G25:2D	1, 2, 3, 4, 21, 30, 46, 56, 71, 82, 96, 108	2	1	1
			G25:2E	1, 2, 3, 4, 21, 30, 46, 56, 71, 82, 96, 109	2	1	1
			G25:2F	1, 2, 11, 16, 24, 37, 44, 50, 63, 91, 119, 120	2	1	1
			G25:2G	1, 3, 5, 7, 9, 28, 43, 54, 65, 84, 97, 104	2	1	1
6	72296	F25:0		1, 3, 4909, 14377, 14385, 19303, 19305, 24259, 32503, 33457, 34069, 34411	0	1	1
12	46664672	F25:0		1, 357511, 2902945, 4638099, 5766395, 9515355, 10313947, 10338407, 12434831, 13091245, 18875539, 21811537	0	2	1?

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Table D.20: (cont.) Circulant graphs of degree 21 to 29 for some diameters k

Degree	k	Order	Iso class	Generating set (plus the involution for odd degree)	Odd girth defect	Maxi- mal levels	Aut group DI
26	6	78597	F26:6	1, 3, 5, 19, 33, 123, 213, 795, 1377, 5139, 8901, 21060, 33219	0	1	1
	6	95024	A26:6	1, 6177, 12471, 15323, 22984, 23092, 24465, 26072, 28373, 30645, 32132, 40277, 41560	0	1	1
27	13	178311348	F27:0	1, 604449, 2176017, 18133357, 19913931, 26247231, 27465939, 45333395, 45333399, 47751195, 53408493, 56455263, 60444525	0	2	1?
29	7	499564	F29:0	1, 3, 10713, 31185, 55815, 74037, 83037, 83045, 108907, 137361, 138841, 138843, 193315, 194611	0	1	1?
	14	911854768	F29:0	1, 4610563, 18116935, 18116969, 91638401, 145565885, 160099805, 178542057, 220564267, 279444447, 291663509, 309042283, 357495113, 366553605	0	2	1?

APPENDIX E

EXTREMAL AND LARGEST-KNOWN BIPARTITE CIRCULANT GRAPHS

E.1 Bipartite circulant graphs up to degree 11

Of the extremal and largest-known bipartite circulant graphs presented in this appendix, the following have been discovered by the author:

Degree 6 and above: diameter 2 and above

For verified extremal graphs the order is shown in bold text. For small diameter, the largest-known graph may have larger order than the member of the largest-known family. Where a graph is a member of an identified largest-known family, the isomorphism class of the family is identified by a code beginning with ‘D’; otherwise the isomorphism class is specific to the graph and begins with ‘E’.

For each known isomorphism class just one generating set is defined: primitive if one exists, otherwise imprimitive. For odd degree, the involutory generator is omitted.

The automorphism group dihedral index (Aut group DI) is the order of the group expressed as a multiple of the order of the dihedral group on the same number of vertices. For large graphs, where the index is conjectured on the basis of the structure of the generating set, this is indicated by a question mark.

Table E.1: Bipartite circulant graphs of degree 2 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Maximal levels	Aut group DI
2	4	D2	1	1	1
3	6	D2	1	2	1
4	8	D2	1	3	1
5	10	D2	1	4	1
6	12	D2	1	5	1
7	14	D2	1	6	1
8	16	D2	1	7	1
9	18	D2	1	8	1
10	20	D2	1	9	1
11	22	D2	1	10	1
12	24	D2	1	11	1
13	26	D2	1	12	1
14	28	D2	1	13	1
15	30	D2	1	14	1
16	32	D2	1	15	1

Table E.2: Bipartite circulant graphs of degree 3 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Maximal levels	Aut group DI
2	6	D3	1	1	6
3	10	D3	1	2	1
4	14	D3	1	3	1
5	18	D3	1	4	1
6	22	D3	1	5	1
7	26	D3	1	6	1
8	30	D3	1	7	1
9	34	D3	1	8	1
10	38	D3	1	9	1
11	42	D3	1	10	1
12	46	D3	1	11	1
13	50	D3	1	12	1
14	54	D3	1	13	1
15	58	D3	1	14	1
16	62	D3	1	15	1

Table E.3: Bipartite circulant graphs of degree 4 for diameter $k \leq 16$

k	Order	Families, D4:s	Generating set parameter, t	Generating set	Maximal levels	Aut group DI
2	8	D4:1, D4:3, D4:5, ...	1	1, 3	1	72
3	18	D4:1, D4:2, D4:4, ...	1	1, 5	2	1
4	32	D4:1, D4:3, D4:5, ...	1	1, 7	3	1
5	50	D4:1, D4:4, D4:6, ...	1	1, 9	4	1
		D4:2, D4:3, D4:7, ...	2	1, 19	4	1
6	72	D4:1, D4:5, D4:7, ...	1	1, 11	5	1
7	98	D4:1, D4:6, D4:8, ...	1	1, 13	6	1
		D4:3, D4:4, ...	2	1, 27	6	1
		D4:2, D4:5, ...	3	1, 41	6	1
8	128	D4:1, D4:7, ...	1	1, 15	7	1
		D4:3, D4:5, ...	3	1, 47	7	1
9	162	D4:1, D4:8, ...	1	1, 17	8	1
		D4:4, D4:5, ...	2	1, 35	8	1
		D4:2, D4:7, ...	4	1, 71	8	1
10	200	D4:1, ...	1	1, 19	9	1
		D4:3, D4:7, ...	3	1, 59	9	1
11	242	D4:1, ...	1	1, 21	10	1
		D4:5, D4:6, ...	2	1, 43	10	1
		D4:4, D4:7, ...	3	1, 65	10	1
		D4:3, D4:8, ...	4	1, 87	10	1
		D4:2, ...	5	1, 109	10	1
12	288	D4:1, ...	1	1, 23	11	1
		D4:5, D4:7, ...	5	1, 119	11	1
13	338	D4:1, ...	1	1, 25	12	1
		D4:6, D4:7, ...	2	1, 51	12	1
		D4:4, ...	3	1, 77	12	1
		D4:3, ...	4	1, 103	12	1
		D4:5, D4:8, ...	5	1, 129	12	1
		D4:2, ...	6	1, 155	12	1
14	392	D4:1, ...	1	1, 27	13	1
		D4:5, ...	3	1, 83	13	1
		D4:3, ...	5	1, 139	13	1
15	450	D4:1, D4:8, ...	1	1, 29	14	1
		D4:7, ...	2	1, 59	14	1
		D4:4, ...	4	1, 119	14	1
		D4:2, ...	7	1, 209	14	1
16	512	D4:1, ...	1	1, 31	15	1
		D4:5, ...	3	1, 95	15	1
		D4:3, ...	5	1, 159	15	1
		D4:7, ...	7	1, 223	15	1

Table E.4: Bipartite circulant graphs of degree 5 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Maximal levels	Aut group DI
2	10	D5	1, 3	1	1440
3	26	D5	1, 5	2	2
4	50	D5	1, 7	3	2
5	82	D5	1, 9	4	2
6	122	D5	1, 11	5	2
7	170	D5	1, 13	6	2
8	226	D5	1, 15	7	2
9	290	D5	1, 17	8	2
10	362	D5	1, 19	9	2
11	442	D5	1, 21	10	2
12	530	D5	1, 23	11	2
13	626	D5	1, 25	12	2
14	730	D5	1, 27	13	2
15	842	D5	1, 29	14	2
16	962	D5	1, 31	15	2

Table E.5: Bipartite circulant graphs of degree 6 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Maximal levels	Aut group DI
2	12	D6:2A/B	1, 3, 5	1	43200
3	38	D6:0	1, 7, 11	2	3
4	80	D6:1A	1, 5, 31	2	2
		D6:1B	1, 11, 15	2	1
5	154	D6:2A	1, 7, 43	3	2
		D6:2B	1, 21, 29	3	1
6	268	D6:0	1, 29, 37	4	3
7	414	D6:1A	1, 9, 91	4	2
		D6:1B	1, 37, 45	4	1
8	616	D6:2A	1, 11, 111	5	2
		D6:2B	1, 55, 67	5	1
9	882	D6:0	1, 67, 79	6	3
10	1196	D6:1A	1, 13, 183	6	2
		D6:1B	1, 79, 91	6	1
11	1590	D6:2A	1, 15, 211	7	2
		D6:2B	1, 105, 121	7	1
12	2072	D6:0	1, 121, 137	8	3
13	2618	D6:1A	1, 17, 307	8	2
		D6:1B	1, 137, 153	8	1
14	3268	D6:2A	1, 19, 343	9	2
		D6:2B	1, 171, 191	9	1
15	4030	D6:0	1, 191, 211	10	3
16	4872	D6:1A	1, 21, 463	10	2
		D6:1B	1, 211, 231	10	1

Table E.6: Bipartite circulant graphs of degree 7 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Maximal levels	Aut group DI
2	14	D7:2	1, 3, 5	1	1814400
3	42	D7:0A	1, 3, 13	1	2
		D7:0B	1, 5, 9	1	1
		E7:3	1, 5, 13	1	1
4	110	D7:1A	1, 5, 21	2	2
		D7:1B	1, 31, 35	2	1
5	234	D7:2	1, 95, 101	3	3
6	406	D7:0A	1, 7, 57	3	2
		D7:0B	1, 83, 91	3	1
7	666	D7:1A	1, 9, 73	4	2
		D7:1B	1, 181, 189	4	1
8	1030	D7:2	1, 459, 469	5	3
9	1474	D7:0A	1, 11, 133	5	2
		D7:0B	1, 329, 341	5	1
10	2054	D7:1A	1, 13, 153	6	2
		D7:1B	1, 547, 559	6	1
11	2786	D7:2	1, 1287, 1301	7	3
12	3630	D7:0A	1, 15, 241	7	2
		D7:0B	1, 839, 855	7	1
13	4658	D7:1A	1, 17, 273	8	2
		D7:1B	1, 1225, 1241	8	1
14	5886	D7:2	1, 2771, 2789	9	3
15	7258	D7:0A	1, 19, 381	9	2
		D7:0B	1, 1709, 1729	9	1
16	8862	D7:1A	1, 21, 421	10	2
		D7:1B	1, 2311, 2331	10	1

Table E.7: Bipartite circulant graphs of degree 8 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Maximal levels	Aut group DI
2	16	D8:0	1, 3, 5, 7	1	101606400
3	58	D8:1	1, 7, 11, 27	1	1
4	160	E8:4A	1, 13, 33, 55	2	1
		E8:4B	1, 15, 25, 77	2	1
5	362	D8:1	1, 61, 75, 131	2	1
6	708	D8:0	1, 7, 97, 133	3	1
7	1298	D8:1	1, 163, 189, 345	3	1
8	2152	D8:0	1, 9, 233, 297	4	1
9	3442	D8:1	1, 345, 387, 723	4	1
10	5160	D8:0	1, 11, 461, 561	5	1
11	7562	D8:1	1, 631, 693, 1313	5	1
12	10596	D8:0	1, 13, 805, 949	6	1
13	14618	D8:1	1, 1045, 1131, 2163	6	1
14	19516	D8:0	1, 15, 1289, 1485	7	1
15	25762	D8:1	1, 1611, 1725, 3321	7	1
16	33168	D8:0	1, 17, 2773, 3097	8	1

Table E.8: Bipartite circulant graphs of degree 9 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Maximal levels	Aut group DI
2	18	E9:2	1, 3, 5, 7	0	7315660800
3	70	E9:3A	1, 7, 11, 19	2	3
		E9:3B	1, 7, 25, 29	2	2
4	198	E9:4	1, 39, 55, 75	1	1
5	482	D9:1	1, 5, 171, 197	2	1
6	1022	D9:0	1, 5, 415, 441	2	1
7	1934	D9:1	1, 7, 785, 835	3	1
8	3362	D9:0	1, 7, 1449, 1499	3	1
9	5474	D9:1	1, 9, 2359, 2441	4	1
10	8462	D9:0	1, 9, 3771, 3853	4	1
11	12542	D9:1	1, 11, 5589, 5711	5	1
12	17954	D9:0	1, 11, 8173, 8295	5	1
13	24962	D9:1	1, 13, 11363, 11533	6	1
14	33854	D9:0	1, 13, 15639, 15809	6	1
15	44942	D9:1	1, 15, 20761, 20987	7	1
16	58562	D9:0	1, 15, 27345, 27571	7	1

Table E.9: Bipartite circulant graphs of degree 10 for diameter $k \leq 16$

k	Order	Iso class	Generating set	Maximal levels	Aut group DI
2	20	D10:2	1, 3, 5, 7, 9	1	658409472000
3	86	D10:3A	1, 7, 11, 19, 35	1	1
		D10:3B	1, 7, 13, 23, 41	1	1
4	288	E10:4	1, 9, 75, 109, 133	2	1
5	714	D10:0	1, 169, 285, 289, 327	2	2
		E10:5	1, 69, 81, 101, 279	3	1
6	1630	D10:1	1, 245, 651, 665, 715	2	2
7	3326	D10:2	1, 107, 557, 647, 1091	3	1
8	6208	D10:3	1, 145, 863, 1031, 2045	3	1
9	10934	D10:4	1, 1757, 2905, 3123, 4655	4	2
10	17940	D10:0	1, 2209, 5127, 5135, 5403	4	2
11	28602	D10:1	1, 2781, 6355, 6705, 9495	4	2
12	43624	D10:2	1, 479, 4367, 4799, 8677	5	1
13	64426	D10:3	1, 581, 5799, 6437, 12827	5	1
14	92818	D10:4	1, 9141, 16115, 16875, 25245	6	2
15	129630	D10:0	1, 10729, 38993, 39875, 39887	6	2
16	178646	D10:1	1, 12597, 27483, 28535, 41145	6	2

Table E.10: Bipartite circulant graphs of degree 11 for diameter $k \leq 16$

k	Order	Iso class	Generating set (plus the involution)	Maximal levels	Aut group DI
2	22	E11:2A	1, 3, 5, 7, 9	1	72425041920000
3	102	E11:3	1, 5, 13, 29, 41	2	1
4	354	E11:4	1, 19, 27, 87, 165	3	1
5	914	D11:0	1, 27, 31, 67, 205	2	1
6	2198	D11:1	1, 53, 207, 563, 827	2	1
7	4658	D11:2	1, 75, 453, 1565, 1939	3	1
8	9102	D11:3	1, 739, 1295, 1301, 3373	3	2
9	16366	D11:4a	1, 503, 4747, 7589, 7595	3	1
		D11:4b	1, 533, 3205, 6077, 7091	3	1
10	28198	D11:0	1, 247, 1983, 10605, 12333	4	1
11	45610	D11:1	1, 313, 2495, 17143, 19959	4	1
12	70486	D11:2	1, 387, 3877, 28233, 31715	5	1
13	106050	D11:3	1, 5251, 19281, 19291, 33219	5	2
14	154154	D11:4a	1, 2287, 11131, 22859, 34001	5	1
		D11:4b	1, 10065, 10077, 33241, 57331	5	1
15	220090	D11:0	1, 827, 9935, 91773, 100869	6	1?
16	305342	D11:1	1, 973, 11663, 127307, 139955	6	1?

APPENDIX F

EXTREMAL AND LARGEST-KNOWN ABELIAN CAYLEY GRAPHS

F.1 Abelian Cayley graphs up to degree 26

Of the extremal and largest-known Abelian Cayley graphs presented in this appendix, the following have been discovered by the author:

Degree 8 and above: diameter 2 and above

Extremal and largest-known Abelian Cayley graphs that are circulant are presented in Appendix D. This appendix only includes undirected extremal and largest-known Abelian Cayley graphs that are non-circulant. In some cases, such as degree 5, non-circulant graphs exist with the same order as the extremal circulant graphs. However, in most cases, the graphs presented in this appendix have larger order than the corresponding circulant graphs.

For small diameter, the largest-known graph may have larger order than the member of the largest-known family. Where a graph is a member of an identified largest-known family, the isomorphism class of the family is identified by a code beginning with ‘A’; otherwise the isomorphism class is specific to the graph and begins with ‘B’.

For each known isomorphism class just one generating set is defined. For odd degree, the involutory generator is included as they are not unique for cyclic rank above 1.

The automorphism group dihedral index (Aut group DI) is the order of the graph’s automorphism group expressed as a multiple of the order of the dihedral group on the same number of vertices. For large graphs, where the index is conjectured on the basis of the structure of the generating set, this is indicated by a question mark.

Table F.1: Non-circulant largest-known Abelian Cayley graphs degree 5

Diameter	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	16	A5	2	4×4	(0 1) (1 0) (2 2)	0	1	60
3	36	A5	2	6×6	(0 1) (1 0) (3 3)	0	2	4
4	64	A5	2	8×8	(0 1) (1 0) (4 4)	0	3	4
5	100	A5	2	10×10	(0 1) (1 0) (5 5)	0	4	4
6	144	A5	2	12×12	(0 1) (1 0) (6 6)	0	5	4
7	196	A5	2	14×14	(0 1) (1 0) (7 7)	0	6	4
8	256	A5	2	16×16	(0 1) (1 0) (8 8)	0	7	4
9	324	A5	2	18×18	(0 1) (1 0) (9 9)	0	8	4
10	400	A5	2	20×20	(0 1) (1 0) (10 10)	0	9	4
11	484	A5	2	22×22	(0 1) (1 0) (11 11)	0	10	4
12	576	A5	2	24×24	(0 1) (1 0) (12 12)	0	11	4
13	676	A5	2	26×26	(0 1) (1 0) (13 13)	0	12	4
14	784	A5	2	28×28	(0 1) (1 0) (14 14)	0	13	4
15	900	A5	2	30×30	(0 1) (1 0) (15 15)	0	14	4
16	1024	A5	2	32×32	(0 1) (1 0) (16 16)	0	15	4

Table F.2: Non-circulant largest-known Abelian Cayley graphs degree 8

Diameter	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	36	B8:2	2	12×3	(0 1) (1 0) (3 1) (5 0)	2	1	8

Table F.3: Non-circulant largest-known Abelian Cayley graphs degree 9

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
4	320	A9:0Aa0	2	40×8	(13 3) (14 3) (14 1) (3 3) (20 4)	0	2	2
		A9:0Ab0	2	40×8	(32 7) (33 7) (39 5) (28 7) (20 4)	0	2	2
		A9:0B0	2	40×8	(27 6) (34 7) (34 1) (43 2) (20 4)	0	2	8
		B9:4A	2	40×8	(1 0) (11 2) (8 3) (8 5) (20 0)	2	3	2
		B9:4B	2	40×8	(1 0) (11 0) (8 3) (8 1) (20 4)	2	3	8
5	724	A9:1	2	362×2	(121 1) (146 1) (218 1) (271 1) (181 0)	0	2	1
6	1440	A9:0Aa2	2	120×12	(43 5) (51 4) (51 2) (97 3) (60 6)	0	3	2
		A9:0Ab2	2	120×12	(93 80) (94 9) (68 3) (87 2) (60 0)	0	3	2
		A9:0B2	2	120×12	(71 8) (83 9) (103 3) (89 2) (60 0)	0	3	8
7	2596	A9:1	2	1298×2	(481 1) (530 1) (722 1) (873 1) (649 0)	0	3	1
8	4352	A9:0Aa0	2	272×16	(105 7) (124 7) (124 1) (71 7) (136 8)	0	4	2
		A9:0Ab0	2	272×16	(216 13) (219 13) (229 11) (192 13) (136 8)	0	4	2
		A9:0B0	2	272×16	(189 12) (212 13) (212 3) (253 4) (136 8)	0	4	8
9	6884	A9:1	2	3442×2	(1361 1) (1442 1) (1842 1) (2171 1) (1721 0)	0	4	1
10	10400	A9:0Aa2	2	520×20	(211 9) (245 8) (245 2) (361 3) (260 10)	0	5	2
		A9:0Ab2	2	520×20	(395 14) (396 15) (334 5) (385 4) (260 0)	0	5	2
		A9:0B2	2	520×20	(339 14) (369 15) (421 5) (389 4) (260 0)	0	5	8
11	15124	A9:1	2	7562×2	(3121 1) (3242 1) (3962 1) (4573 1) (3781 0)	0	5	1
12	21312	A9:0Aa0	2	888×24	(373 11) (426 11) (426 1) (299 11) (444 12)	0	6	2
		A9:0Ab0	2	888×24	(696 19) (701 19) (715 17) (636 19) (444 12)	0	6	2
		A9:0B0	2	888×24	(631 18) (678 19) (678 5) (775 6) (444 12)	0	6	8
13	29236	A9:1	2	14618×2	(6217 1) (6386 1) (7562 1) (8583 1) (7309 0)	0	6	1
14	39200	A9:0Aa2	2	1400×28	(603 13) (679 12) (679 2) (897 3) (700 14)	0	7	2
		A9:0Ab2	2	1400×28	(1057 20) (1058 21) (944 7) (1043 6) (700 0)	0	7	2
		A9:0B2	2	1400×28	(951 20) (1007 21) (1107 7) (1049 6) (700 0)	0	7	8
15	51524	A9:1	2	25762×2	(11201 1) (11426 1) (13218 1) (14801 1) (12881 0)	0	7	1
16	66560	A9:0Aa0	2	2080×32	(913 15) (1016 15) (1016 1) (783 15) (1040 16)	0	8	2
		A9:0Ab0	2	2080×32	(1616 25) (1623 25) (1641 23) (1504 25) (1040 16)	0	8	2
		A9:0B0	2	2080×32	(1497 24) (1576 25) (1576 7) (1753 8) (1040 16)	0	8	8

Table F.4: Non-circulant largest-known Abelian Cayley graphs degree 10

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
6	2425	A10:1	2	485×5	(177 1) (25 4) (25 1) (17 1) (35 0)	0	3	4
7	4644	A10:2	3	258×6×3	(164 2 1) (29 0 0) (29 1 0) (72 3 2) (80 2 1)	0	3	6
8	8477	A10:3	2	1211×7	(327 1) (49 6) (49 1) (19 1) (35 0)	0	4	4
11	35883	A10:1	2	3987×9	(857 1) (81 8) (81 1) (29 1) (99 0)	0	5	4
12	53500	A10:2	3	1070×10×5	(590 4 2) (71 0 0) (71 1 0) (178 5 3) (230 4 2)	0	5	6
13	78287	A10:3	2	7117×11	(1263 1) (121 10) (121 1) (31 1) (99 0)	0	6	4
16	209053	A10:1	2	16081×13	(2433 1) (169 12) (169 1) (41 1) (195 0)	0	7	4?

Table F.5: Non-circulant largest-known Abelian Cayley graphs degree 11

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	56	B11:2	2	28×2	(2 0) (4 1) (6 0) (7 0) (7 1) (14 1)	2	1	4
4	576	A11:4	2	192×3	(97 1) (45 1) (45 2) (31 1) (9 0) (96 0)	0	2	4
5	1472	A11:0	3	92×4×4	(35 3 3) (21 1 1) (9 3 0) (9 0 3) (9 0 0) (46 2 2)	0	2	6
6	3400	A11:1	2	680×5	(479 4) (555 4) (555 1) (609 4) (605 0) (340 0)	0	3	4
9	22148	A11:4	2	3164×7	(613 1) (441 1) (441 6) (291 1) (245 0) (1582 0)	0	4	4
10	36352	A11:0	3	568×8×8	(99 7 7) (77 1 1) (49 7 0) (49 0 7) (49 0 0) (284 4 4)	0	4	6
11	57996	A11:1	2	6444×9	(5503 8) (5715 8) (5715 1) (5953 8) (5877 0) (3222 0)	0	5	4
14	186824	A11:4	2	16984×11	(1929 1) (1573 1) (1573 10) (1159 1) (1089 0) (8492 0)	0	6	4?
15	260928	A11:0	3	1812 x12 x12	(195 11 11) (165 1 1) (121 11 0) (121 0 11) (121 0 0) (906 6 6)	0	6	6?
16	359632	A11:1	2	27664×13	(25055 12) (25467 12) (25467 1) (26017 1) (25805 0) (13832 0)	0	7	4?

Table F.6: Non-circulant largest-known Abelian Cayley graphs degree 12

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	72	B12:2A	2	12×6	(0 1) (2 0) (2 1) (3 0) (3 3) (4 4)	2	1	24
		B12:2B		36×2	(2 0) (4 1) (9 0) (9 1) (10 1) (12 0)	2	1	12
5	2128	A12:2	2	532×4	(374 0) (91 0) (39 1) (94 1) (3 0) (306 0)	0	2	2
6	5200	A12:0	2	1300×4	(38 0) (93 0) (14 1) (311 1) (107 0) (182 0)	0	3	2
7	10900	A12:1a	2	2180×5	(430 0) (100 0) (705 0) (247 1) (247 0) (189 1)	0	3	6
		A12:1b	2	2180×5	(750 0) (510 0) (95 0) (67 4) (939 0) (939 1)	0	3	6
8	21780	A12:2	2	3630×6	(1587 0) (328 0) (69 1) (536 1) (208 0) (1293 0)	0	3	2
9	40788	A12:0	2	6798×6	(129 0) (278 0) (16 5) (1149 1) (262 0) (765 0)	0	4	2
10	70756	A12:1a	2	10108×7	(1526 0) (350 0) (2555 0) (633 1) (633 0) (811 1)	0	4	6
		A12:1b	2	10108×7	(2856 0) (1246 0) (707 0) (331 6) (3219 0) (3219 1)	0	4	6
11	119104	A12:2	2	14888×8	(4636 0) (821 0) (75 1) (1786 1) (965 0) (3892 0)	0	4	2
12	192832	A12:0	2	24104×8	(300 0) (619 0) (102 7) (3115 1) (517 0) (2252 0)	0	5	2?
13	298404	A12:1a	2	33156×9	(3978 0) (828 0) (6813 0) (1291 1) (1291 0) (2393 1)	0	5	6?
		A12:1b	2	33156×9	(7974 0) (2466 0) (2367 0) (875 8) (8243 0) (8243 1)	0	5	6?
14	452500	A12:2	2	45250×10	(10865 0) (1666 0) (33 1) (4492 1) (2826 0) (9375 0)	0	5	2?
15	668500	A12:0	2	66850×10	(575 0) (1164 0) (268 9) (6953 1) (896 0) (5315 0)	0	6	2?
16	958804	A12:1a	2	87164×11	(8602 0) (1606 0) (15015 0) (2293 1) (2293 0) (5631 1)	0	6	6?
		A12:1b	2	87164×11	(18216 0) (4290 0) (5819 0) (1795 10) (17643 0) (17643 1)	0	6	6?

Table F.7: Non-circulant largest-known Abelian Cayley graphs degree 13

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	80	B13:2	2	40×2	(2 0) (5 0) (5 1) (6 1) (14 1) (18 0) (20 1)	0	1	128
6	6656	A13:0a	3	$416 \times 4 \times 4$	(103 1 3) (71 1 0) (14 1 0) (14 3 3) (57 3 0) (129 3 1) (208 0 2)	0	2	2
		A13:0b	3	$416 \times 4 \times 4$	(95 1 0) (51 1 3) (48 3 0) (56 3 1) (109 3 3) (39 1 0) (0 0 2)	0	2	2
6	7168	A13:0c	4	$56 \times 8 \times 4 \times 4$	(36 5 3 3) (47 6 3 0) (53 7 3 3) (53 7 1 0) (47 0 1 1) (6 7 3 0) (28 4 0 2)	0	3	24
7	15200	A13:1a1	2	760×20	(235 5) (45 5) (110 15) (74 3) (74 19) (78 17) (0 10)	0	3	12
		A13:1b1	2	760×20	(105 5) (85 5) (110 5) (178 13) (178 5) (126 3) (380 10)	0	3	12
8	31200	A13:2a2	2	1560×20	(655 5) (395 15) (480 5) (318 19) (318 15) (306 9) (780 0)	0	3	12
		A13:2b2	2	1560×20	(5 5) (265 5) (75 0) (381 18) (381 14) (693 18) (0 10)	0	3	12
9	59616	A13:0a	3	$1656 \times 6 \times 6$	(301 1 5) (239 1 0) (106 1 0) (106 5 5) (133 5 0) (347 5 1) (828 0 3)	0	3	2
		A13:0b	3	$1656 \times 6 \times 6$	(275 1 0) (161 1 5) (84 5 0) (192 5 1) (307 5 5) (83 1 0) (0 0 3)	0	3	2
9	62208	A13:0d	4	$72 \times 24 \times 6 \times 6$	(19 7 3 3) (38 13 4 4) (55 18 4 5) (55 18 2 2) (38 17 2 1) (65 1 3 2) (36 12 0 3)	0	4	24
10	108192	A13:1a4	2	3864×28	(1645 7) (1001 7) (903 0) (597 2) (597 22) (45 18) (0 14)	0	4	12
		A13:1b4	2	3864×28	(679 7) (35 21) (798 7) (1320 9) (1320 5) (216 21) (1932 0)	0	4	12
11	185024	A13:2a5	2	6608×28	(2177 7) (1351 21) (1736 7) (1136 7) (1136 23) (192 19) (0 14)	0	4	12?
		A13:2b5	2	6608×28	(539 7) (1365 7) (1036 21) (2308 9) (2308 13) (3252 13) (0 14)	0	4	12?
12	303104	A13:0a	3	$4736 \times 8 \times 8$	(659 1 7) (559 1 0) (318 1 0) (318 7 7) (241 7 0) (733 7 1) (2368 0 4)	0	4	2?

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Table F.7: (cont.) Non-circulant largest-known Abelian Cayley graphs
degree 13

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
		A13:0b	3	$4736 \times 8 \times 8$	(607 1 0) (391 1 7) (128 7 0) (464 7 1) (665 7 7) (143 1 0) (0 0 4)	0	4	2?
12	311296	A13:0c	4	$304 \times 16 \times 8 \times 8$	(144 9 4 5) (189 11 5 5) (235 12 6 5) (235 14 7 3) (189 3 6 3) (274 7 7 5) (152 8 4 0)	0	5	24?
13	482112	A13:1a1	2	13392×36	(4221 9) (2547 9) (2700 27) (1924 7) (1924 3) (436 35) (0 18)	0	5	12?
		A13:1b1	2	13392×36	(2097 9) (423 27) (2412 9) (3524 33) (3524 1) (548 29) (6696 18)	0	5	12?
14	738720	A13:2a2	2	20520×36	(7731 9) (5679 27) (6390 9) (4640 31) (4640 27) (80 23) (10260 0)	0	5	12?
		A13:2b2	2	20520×36	(1899 9) (3951 9) (3105 0) (5555 34) (5555 26) (7835 30) (0 18)	0	5	12?
15	1100000	A13:0a	3	$11000 \times 10 \times 10$	(1225 1 9) (1079 1 0) (698 1 0) (698 9 9) (381 9 0) (1335 9 1) (5500 0 5)	0	5	2?
		A13:0b	3	$11000 \times 10 \times 10$	(1139 1 0) (789 1 9) (180 9 0) (920 9 1) (1231 9 9) (219 1 0) (0 0 5)	0	5	2?
15	1120000	A13:0d	4	$280 \times 40 \times 10 \times 10$	(129 19 5 5) (163 24 6 6) (202 29 8 8) (202 29 8 9) (163 28 8 7) (235 1 9 8) (140 0 0 5)	0	6	24?
16	1606880	A13:1a4	2	36520×44	(13651 11) (9999 11) (9405 0) (6955 2) (6955 34) (3635 30) (0 22)	0	6	12?
		A13:1b4	2	36520×44	(7271 11) (3619 33) (8360 11) (11710 13) (11710 9) (5070 33) (18260 0)	0	6	12?
18	3234816	A13:0c	4	$936 \times 24 \times 12 \times 12$	(734 19 10 10) (775 20 10 1) (821 21 11 10) (821 21 1 11) (775 22 2 2) (904 1 10 1) (468 12 0 6)	0	7	24?

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Table F.7: (cont.) Non-circulant largest-known Abelian Cayley graphs
degree 13

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
21	7990528	A13:0d	4	$728 \times 56 \times 14 \times 14$	(597 47 12 12) (620 49 12 13) (647 50 13 12) (647 2 1 1) (620 53 2 2) (703 9 12 13) (364 28 0 7)	0	8	24?

Table F.8: Non-circulant largest-known Abelian Cayley graphs degree 14

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
2	90	B14:2	2	30×3	(1 0) (3 0) (4 0) (5 1) (5 2) (12 0) (15 1)	2	1	18
5	3717	A14:5a	2	1239×3	(641 1) (213 0) (81 2) (81 1) (159 0) (185 1) (21 0)	0	2	4
		A14:5b	2	1239×3	(425 1) (261 0) (921 1) (921 2) (777 0) (401 1) (123 0)	0	2	4
6	10096	A14:6	2	2524×4	(700 0) (180 0) (241 1) (241 0) (390 1) (152 0) (68 0)	2	3	6
7	23504	A14:0	2	5876×4	(780 0) (316 0) (99 1) (99 0) (1370 1) (488 0) (372 0)	0	3	6
8	50575	A14:1a	2	10115×5	(3503 1) (165 0) (625 1) (625 4) (415 0) (543 1) (885 0)	0	3	4
		A14:1b	2	10115×5	(4503 4) (3795 0) (2295 4) (2295 1) (5215 0) (457 1) (885 0)	0	3	4
8	50800	A14:1c	2	2540×20	(2460 0) (2499 2) (55 2) (55 18) (1991 18) (120 15) (2025 15)	0	3	8
9	105300	A14:2	2	3510×30	(516 6) (3330 6) (3330 24) (186 6) (5 5) (5 0) (2930 25)	0	4	24
11	354564	A14:4	2	8442×42	(1212 0) (6 0) (252 36) (252 6) (287 7) (287 0) (1120 7)	0	4	24?
12	602063	A14:5a	2	86009×7	(22633 1) (3885 0) (2401 6) (2401 1) (3199 0) (1941 1) (1701 0)	0	2	4?
		A14:5b	2	86009×7	(17145 1) (17745 0) (28497 1) (28497 6) (23261 0) (7429 1) (525 0)	0	2	4?
12	608384	A14:5c	2	21728×28	(19684 14) (18562 5) (20314 9) (20314 5) (15458 9) (1533 0) (20545 7)	0	4	8?
13	1010752	A14:6	2	126344×8	(18024 0) (3544 0) (3123 1) (3123 0) (12670 1) (3056 0) (360 0)	0	5	6?

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Table F.8: (cont.) Non-circulant largest-known Abelian Cayley graphs
degree 14

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
14	1617344	A14:0	2	202168×8	(18344 0) (4584 0) (523 7) (523 0) (25794 1) (5648 0) (3928 0)	0	5	6?
15	2509785	A14:1a	2	278865×9	(58615 1) (3789 0) (6561 1) (6561 8) (5247 0) (3355 1) (8037 0)	0	5	4?
		A14:1b	2	278865×9	(70279 8) (62559 0) (46287 8) (46287 1) (71667 0) (8309 1) (8685 0)	0	5	4?
15	2529792	A14:1c	2	70272×36	(26244 18) (66526 1) (68598 1) (68598 17) (58718 17) (2331 27) (63819 18)	0	5	8?
16	3879900	A14:2	2	43110×90	(4060 0) (730 0) (900 80) (900 10) (153 81) (153 0) (4464 9)	0	4	24?
22	31580016	A14:1c	2	607308×52	(251472 0) (583437 1) (592917 1) (592917 25) (536721 25) (10270 39) (566969 0)	0	7	8?

Table F.9: Non-circulant largest-known Abelian Cayley graphs degree 15

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maximal levels	Aut group DI
2	100	B15:2	2	20×5	(0 1) (0 2) (1 0) (3 1) (5 0) (7 3) (9 4) (10 0)	2	1	4
6	12672	A15:6	3	264×12 ×4	(139 2 0) (63 2 0) (63 10 0) (37 2 0) (9 0 1) (9 9 0) (9 9 3) (132 0 0)	0	2	24
7	31744	A15:0	5	124×4 ×4×4×4	(42 1 1 1 1) (18 1 1 0 0) (18 3 3 0 3) (18 3 3 3 0) (7 1 1 1 1) (7 1 0 1 1) (100 3 0 3 3) (0 2 2 2 2)	0	3	36
8	70400	A15:1	3	880×20 ×4	(291 2 0) (175 2 0) (175 18 0) (61 2 0) (75 0 1) (75 15 0) (75 15 3) (440 0 0)	0	3	24
9	146400	A15:2	3	7320×10 ×2	(3615 0 0) (5427 4 1) (2535 4 1) (2535 6 1) (3963 6 1) (2470 5 0) (1250 5 1) (0 0 1)	0	3	8
10	287712	A15:3	3	23976×6 ×2	(5451 3 1) (10101 3 1) (14601 3 1) (15501 3 1) (20363 3 0) (20363 4 1) (4379 2 0) (11988 0 0)	0	4	6?
11	526608	A15:4	3	43884×6 ×2	(37137 3 1) (33783 3 1) (30507 3 1) (30975 3 1) (14753 3 0) (14753 2 1) (125 4 0) (21942 0 0)	0	4	6?
12	929040	A15:5	3	33180 ×14×2	(24507 7 1) (21543 6 1) (12495 6 1) (12495 8 1) (16803 8 1) (11564 0 1) (6034 7 1) (0 7 0)	0	4	8?
13	1593088	A15:6	3	14224 ×28×4	Not found			
14	2588672	A15:0	5	632×8 ×8×8×8	(121 1 1 1 1) (77 1 1 0 0) (77 7 7 0 7) (77 7 7 7 0) (35 1 0 1 1) (35 0 1 1 1) (35 0 0 7 7) (316 4 4 4)	0	5	36?
15	4084992	A15:1	3	28368 ×36×4	Not found			
16	6266160	A15:2	3	174060 ×18×2	(56205 9 1) (43115 10 1) (70515 10 1) (70515 8 1) (62455 8 1) (94284 0 1) (129096 9 1) (87030 9 0)	0	5	8?

Table F.10: Non-circulant largest-known Abelian Cayley graphs degree 18

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maximal levels	Aut group DI
2	147	B18:2	2	21×7	(0 1) (1 0) (1 3) (4 1) (4 4) (5 2) (5 6) (7 3) (7 4)	2	1	6

Table F.11: Non-circulant largest-known Abelian Cayley graphs degree 19

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
9	704512	A19:0	7	172×4	(49 0 0 1 1 1 1) (49 0 3 3 3 3 3) ×4×4 (49 3 0 3 3 3 3) (7 3 3 0 0 3 3) ×4×4 (7 1 1 0 1 1 1) (7 1 1 1 0 1 1) ×4 (1 1 1 1 1 0 0) (1 3 3 3 3 0 3) (1 3 3 3 3 3 0) (86 2 2 2 2 2 2)	0	3	9?
18	190840832	A19:0	7	728×8	(121 0 0 1 1 1 1) (121 0 7 7 7 7 7) ×8×8 (121 7 0 7 7 7 7) (55 7 7 0 0 7 7) ×8×8 (55 1 1 0 1 1 1) (55 1 1 1 0 1 1) ×8 (25 1 1 1 1 0 0) (25 7 7 7 7 0 7) (25 7 7 7 7 7 0) (364 4 4 4 4 4 4)	0	6	9?

Table F.12: Non-circulant largest-known Abelian Cayley graphs degree 21

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
5	10464	A21:0	2	2616×4	(2189 3) (850 1) (912 1) (1211 1) (1036 1) (532 3) (1047 1) (2264 3) (1398 3) (2093 3) (0 2)	0	1	1
10	2388096	A21:0	2	298512×8	(108235 2) (66708 1) (67720 7) (75543 0) (71768 7) (66320 1) (65503 2) (48576 1) (53012 7) (273115 0) (149256 4)	0	2	1?
15	90549216	A21:0	2	7545768×12	(1648003 2) (1181712 1) (1160538 11) (1213681 10) (1033494 11) (1045074 1) (7530109 2) (987510 1) (997104 11) (5957371 10) (3772884 6)	0	3	1?

Table F.13: Non-circulant largest-known Abelian Cayley graphs degree 26

Diam	Order	Iso class	Cyclic rank	Cyclic order	Generating set	Odd girth defect	Maxi- mal levels	Aut group DI
6	95024	A26:6	1*	95024×1*	1, 6177, 12471, 15323, 22984, 23092, 24465, 26072, 28373, 30645, 32132, 40277, 41560	0	1	1

* Cyclic rank of this family is generally 2, but is 1 for diameter 6
as the second cyclic order is 1 in this case

APPENDIX G

DIRECTED AND MIXED CIRCULANT GRAPHS

Of the extremal and largest-known directed and mixed circulant graphs presented in this appendix, the following have been discovered by the author:

Directed circulant graphs

Directed degree 3: diameter 44 to 48

Directed degree 4: diameter 2 to 22

Directed degree 5: diameter 2 to 9

Mixed circulant graphs of dimension 2

Directed degree 2, undirected degree 1: diameter 2 and above
(also independently by C. Dalfó, M.A. Fiol and N. López [8])

Directed degree 1, undirected degree 2: diameter 2 and above

Directed degree 1, undirected degree 3: diameter 2 and above

Mixed circulant graphs of dimension 3

Directed degree 3, undirected degree 1: diameter 2 to 37

Directed degree 2, undirected degree 2: diameter 2 to 45

Directed degree 2, undirected degree 3: diameter 2 to 37

Directed degree 1, undirected degree 4: diameter 2 to 49

Directed degree 1, undirected degree 5: diameter 2 to 37

Mixed circulant graphs of dimension 4

Directed degree 3, undirected degree 2: diameter 2 to 17

Directed degree 2, undirected degree 4: diameter 2 to 15

Directed degree 1, undirected degree 6: diameter 2 to 14

For verified extremal graphs the order is shown in bold text. Where existence has not been checked up to the upper bound, the limit of checking is stated in the column *Checked up to*.

For small diameter, the largest-known graph may have larger order than the member of the largest-known family. Where a graph is a member of an identified largest-known family, the isomorphism class of the family is identified by a code beginning with ‘H’ for directed graphs and with ‘M’ for mixed graphs. No code is given to other families or graphs. For each known isomorphism class just one generating set is defined.

Directed and mixed circulant graphs have rotational symmetry, but, in contrast to undirected graphs, have no reflexive symmetry. So their minimum possible automorphism group is the cyclic group on the vertices instead of the dihedral group. It is therefore convenient to express the size of the automorphism group as a multiple of the size of the cyclic group, called the *cyclic index* (Aut group CI).

G.1 Directed circulant graphs

Table G.1: Directed circulant graphs of directed degree 2

Diameter k	Order	Isomorphism class	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI
2	5	H2:2B	1, 2	3	3	1	1
3	8	H2:0A	1, 6	3	3	2	1
		H2:0B	1, 3	bipartite	4	2	2
4	11	H2:1A	1, 5	3	3	2	1
		H2:1B	1, 8	5	4	2	1
		H2:1C	1, 4	5	5	2	1
5	16	H2:2A	1, 7	bipartite	4	3	2
		H2:2B	1, 10	5	5	3	1
6	21	H2:0A	1, 9	5	5	4	1
		H2:0B	1, 13	9	6	4	2
7	26	H2:1A	1, 8	5	5	4	1
		H2:1B	1, 11	bipartite	6	4	1
		H2:1C	1, 16	7	7	4	1
8	33	H2:2A	1, 10	15	6	5	2
		H2:2B	1, 24	7	7	5	1
9	40	H2:0A	1, 12	7	7	6	1
		H2:0B	1, 29	bipartite	8	6	2
10	47	H2:1A	1, 11	7	7	6	1
		H2:1B	1, 14	11	8	6	1
		H2:1C	1, 34	9	9	6	1
11	56	H2:2A	1, 13	bipartite	8	7	2
		H2:2B	1, 44	9	9	7	1
12	65	H2:0A	1, 15	9	9	8	1
		H2:0B	1, 51	15	10	8	2
13	74	H2:1A	1, 14	9	9	8	1
		H2:1B	1, 17	bipartite	10	8	1
		H2:1C	1, 58	11	11	8	1
14	85	H2:2A	1, 16	25	10	9	2
		H2:2B	1, 70	11	11	9	1
15	96	H2:0A	1, 18	11	11	10	1
		H2:0B	1, 79	bipartite	12	10	2
16	107	H2:1A	1, 17	11	11	10	1
		H2:1B	1, 20	17	12	10	1
		H2:1C	1, 88	13	13	10	1

Table G.2: Directed circulant graphs of directed degree 3

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
2	9	1, 3, 4	3	3	1	1	10
		1, 4, 6	3	3	1	1	
3	16	1, 4, 5	5	4	1	1	20
		1, 5, 12	5	4	1	1	
4	27	1, 4, 17	5	5	2	1	35
		1, 5, 12	5	4	2	1	
		1, 6, 8	5	5	2	1	

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Table G.2: (cont.) Directed circulant graphs of directed degree 3

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
5	40	1, 16, 23	5	5	2	1	56
		1, 6, 15	7	6	3	1	
		1, 6, 25	7	6	3	1	
		1, 16, 35	5	5	3	1	
		1, 26, 35	5	5	3	1	
6	57	1, 13, 33	7	6	2	1	84
		1, 16, 36	7	7	2	1	
7	84	2, 9, 35	9	8	3	1	120
8	111	1, 31, 69	9	9	5	1	159
9	138	1, 11, 78	9	9	5	1	207
		1, 17, 96	9	9	5	1	
		1, 19, 26	9	9	5	1	
		1, 43, 122	9	9	5	1	
10	176	1, 17, 56	9	9	5	1	263
		1, 24, 33	11	11	5	1	
		1, 32, 153	11	10	5	1	
		1, 41, 64	9	9	5	1	
		1, 81, 104	9	9	5	1	
		1, 121, 160	9	9	5	1	
11	217	1, 13, 119	11	11	5	1	329
		1, 18, 46	13	12	5	1	
		1, 34, 161	11	10	5	1	
		1, 51, 92	11	11	5	1	
12	273	1, 14, 153	15	12	6	1	405
		1, 49, 104	11	11	6	1	
		1, 53, 186	11	11	6	1	
		1, 88, 221	11	11	6	1	
13	340	1, 90, 191	17	14	7	1	491
14	395	1, 35, 271	17	12	6	1	589
		1, 125, 361	15	15	6	1	
15	462	1, 29, 97	bipartite	16	7	1	699
		1, 33, 254	13	13	7	1	
		1, 44, 56	15	15	8	1	
		1, 44, 408	17	12	8	1	
		1, 55, 419	bipartite	14	8	1	
		1, 89, 121	bipartite	12	8	1	
		1, 110, 254	15	12	8	1	
		1, 122, 165	13	12	7	1	
		1, 165, 188	17	12	7	1	
		1, 209, 430	19	16	7	1	
		1, 224, 380	15	15	7	1	
		1, 275, 298	13	13	7	1	
		1, 282, 296	15	15	7	1	
		1, 298, 341	13	13	7	1	
		1, 342, 374	15	12	8	1	
		1, 366, 434	13	13	7	1	
		2, 253, 354	13	13	8	1	
		6, 28, 143	13	13	8	1	
16	560	1, 215, 326	15	15	9	1	823
		1, 235, 346	21	16	9	1	
17	648	1, 76, 237	21	18	7	1	960
		1, 412, 573	15	14	7	1	

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Table G.2: (cont.) Directed circulant graphs of directed degree 3

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
18	748	1, 41, 147 1, 174, 362 1, 490, 676 1, 602, 708	bipartite 17 21 15	16 17 18 15	7 7 7 7	1 1 1 1	1111
19	861	1, 27, 463 1, 84, 298 1, 84, 319 1, 543, 778	17 19 15 25	17 18 15 20	8 8 8 8	1 1 1 1	1277
20	979	1, 22, 351 1, 138, 787 1, 193, 842 1, 374, 637	19 17 17 23	19 17 17 16	10 10 10 10	1 1 1 1	1460
21	1140	1, 45, 196 1, 945, 1096	27 31	22 20	11 11	1 1	1658
22	1305	1, 246, 1030 1, 276, 1060	21 19	21 19	9 9	1 1	1875
23	1440	1, 126, 415 1, 1026, 1315	21 23	21 20	9 9	1 1	2109
24	1616	1, 56, 257 1, 416, 617	23 33	23 20	10 10	1 1	2361
25	1788	1, 154, 1452 1, 192, 1174 1, 337, 1635 2, 267, 818	21 29 bipartite 29	21 22 24 24	14 14 14 14	1 1 1 1	2634
26	1963	1, 90, 780 1, 142, 1014 1, 169, 1594 1, 222, 1657 1, 236, 768 1, 307, 1742 1, 341, 887 1, 397, 891	27 23 23 25 21 29 31 21	22 23 23 25 21 24 22 21	12 12 14 14 12 14 14 12	1 1 1 1 1 1 1 1	2926
27	2224	1, 425, 704 1, 1025, 1304	25 29	25 24	15 15	1 1	3240
28	2442	1, 964, 1372 1, 1071, 1479 2, 285, 1752 2, 1185, 1590	27 bipartite 23 25	24 26 23 25	11 11 11 11	1 1 1 1	3574
29	2693	1, 39, 942 1, 161, 1676 1, 373, 2259 1, 435, 2321	25 31 23 35	25 28 23 26	12 12 12 12	1 1 1 1	3932
30	2920	1, 540, 831 1, 890, 1181	27 25	27 25	12 12	1 1	4312
31	3220	7, 30, 2277	27	27	13	1	4716
32	3591	1, 1519, 2031 1, 1561, 2073	29 27	29 27	13 13	1 1	5145
33	3850	2, 475, 1177 2, 777, 1325 2, 2527, 3075 2, 2675, 3377	29 29 35 35	29 29 28 28	13 13 13 13	1 1 1 1	5598

continued on next page

Table G.2: (cont.) Directed circulant graphs of directed degree 3

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
34	4191	1, 748, 2652 1, 1540, 3444	31 39	31 28	14 14	1 1	6078
35	4468	1, 353, 2789 1, 444, 2360 1, 480, 3772 1, 1680, 4116	bipartite 33 39 37	32 30 28 30	14 14 14 14	1 1 1 1	6584
36	4871	1, 238, 1113 1, 781, 3437 1, 853, 2768 1, 1435, 4091	33 47 33 31	33 30 33 31	20 20 20 20	1 1 1 1	7118
37	5328	1, 345, 2344 1, 2985, 4984	33 39	30 34	15 15	1 1	7680
38	5698	1, 1375, 2410 1, 3289, 4324	33 45	33 34	21 21	1 1	8270
39	6131	1, 51, 1589 1, 277, 2575 1, 333, 2692 1, 1172, 1684	33 37 31 45	33 34 31 36	16 16 16 16	1 1 1 1	8890
40	6513	1, 560, 5070 1, 1444, 5954	33 35	33 35	16 16	1 1	8890
41	6942	1, 793, 1860 1, 5083, 6150	41 45	38 32	17 17	1 1	8890
42	7533	1, 1612, 4961 1, 1612, 5798 1, 1736, 5922 1, 2573, 5922	45 41 35 41	36 36 35 36	17 17 17 17	1 1 1 1	8890
43	8064	1, 1377, 4960 1, 3105, 6688	41 37	36 37	17 17	1 1	8890
44	8567	1, 57, 6904 1, 154, 6150 1, 1550, 3627 1, 1664, 8511	37 47 41 47	37 34 41 38	18 18 18 18	1 1 1 1	8890
45	9070	1, 695, 5735 1, 3336, 8376 2, 780, 3425 2, 2730, 6615	bipartite 45 59 47	38 40 36 38	18 18 18 18	1 1 1 1	9300
46	9685	1, 2042, 6513	53	40	25	1	9900
47	10340	1, 1961, 7346 1, 2995, 8380 1, 5200, 6235 2, 1105, 4205	55 41 39 39	40 41 39 39	19 19 19 19	1 1 1 1	10520
48	10990	1, 4786, 5886 1, 5105, 6205 2, 5435, 8730 2, 9000, 9685	45 bipartite 41 39	40 42 41 39	19 19 19 19	1 1 1 1	11120

Table G.3: Directed circulant graphs of directed degree 4

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
2	13	1, 2, 6, 9	3	3	1	1	15
3	25	1, 2, 8, 19	5	4	1	1	35
4	49	1, 3, 12, 20	5	5	2	1	70
		1, 3, 32, 37	5	5	2	1	
		1, 5, 13, 32	5	5	2	1	
		1, 13, 18, 47	3	3	2	1	
		1, 16, 20, 26	5	4	2	1	
5	83	1, 4, 20, 50	7	6	2	1	126
		1, 4, 62, 67	5	5	2	1	
		1, 5, 27, 63	5	5	2	1	
		1, 5, 34, 62	5	5	2	1	
		1, 17, 22, 80	7	4	2	1	
6	130	1, 11, 46, 52	5	5	3	1	210
		1, 60, 79, 115	5	5	3	2	
		1, 79, 85, 120	7	6	3	2	
		2, 28, 75, 115	9	4	3	2	
7	196	1, 5, 87, 166	7	7	3	1	330
		1, 7, 35, 114	7	7	4	1	
		1, 7, 40, 62	7	6	3	1	
		1, 7, 74, 84	7	6	4	1	
		1, 7, 79, 146	7	6	3	1	
		1, 10, 46, 123	7	7	3	1	
		1, 19, 120, 130	5	5	3	1	
		1, 51, 79, 86	7	7	4	1	
		1, 83, 162, 190	7	7	4	1	
		1, 112, 119, 130	5	5	4	1	
8	277	1, 7, 40, 170	9	8	4	1	495
		1, 7, 158, 175	9	6	4	1	
		1, 18, 80, 253	7	6	4	1	
		1, 29, 46, 257	7	7	4	1	
		1, 46, 145, 259	7	7	4	1	
9	390	1, 34, 155, 217	7	7	4	1	715
		1, 56, 159, 258	9	9	4	1	
		1, 153, 177, 206	9	9	4	1	
		1, 174, 236, 357	7	7	4	1	
		1, 185, 214, 238	7	7	4	1	
10	536	1, 40, 149, 355	9	9	5	1	1001
		1, 50, 190, 401	7	7	5	1	
		1, 56, 363, 437	9	9	5	1	
		1, 100, 174, 481	9	8	5	1	
		1, 177, 260, 290	13	6	5	1	
11	710	1, 34, 219, 334	11	10	5	1	1365
		1, 43, 566, 591	9	9	5	1	
		1, 120, 145, 668	11	10	5	1	
		1, 180, 495, 598	9	8	5	1	
		1, 202, 395, 445	13	10	5	1	
12	922	1, 12, 247, 384	11	11	5	1	1820
		1, 50, 504, 733	11	8	5	1	
		1, 66, 208, 607	11	11	5	1	
		1, 190, 419, 873	11	11	5	1	
		1, 311, 362, 533	13	10	5	1	
13	1161	1, 15, 497, 734	11	11	6	1	2380

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Table G.3: (cont.) Directed circulant graphs of directed degree 4

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
14	1451	1, 65, 94, 608	13	12	6	1	2400
		1, 71, 644, 849	11	11	6	1	
		1, 79, 176, 1014	13	12	6	1	
		1, 84, 629, 695	11	11	6	1	
		1, 148, 986, 1083	11	11	6	1	
		1, 179, 205, 236	11	10	6	1	
		1, 201, 463, 1034	11	11	6	1	
		1, 221, 317, 573	17	10	6	1	
		1, 249, 437, 634	13	13	6	1	
		1, 12, 795, 1143	13	13	6	1	
		1, 109, 264, 1309	17	14	6	1	
		1, 116, 546, 828	13	13	7	1	
		1, 127, 721, 782	11	11	6	1	
		1, 140, 322, 733	11	11	6	1	
		1, 143, 1188, 1343	19	10	6	1	
15	1800	1, 117, 331, 1054	17	14	7	1	2400
		1, 459, 532, 1222	17	14	7	1	
		1, 579, 1269, 1342	13	13	7	1	
		1, 747, 1470, 1684	15	10	7	1	
		1, 1167, 1230, 1594	13	13	7	1	
16	2255	1, 13, 202, 359	13	13	8	1	2400
17	2723	1, 23, 1548, 1782	15	15	7	1	2850
		1, 140, 1269, 2035	13	13	7	1	
		1, 147, 636, 1488	17	17	7	1	
		1, 302, 1034, 1115	23	12	7	1	
		1, 636, 764, 857	17	12	7	1	
18	3264	1, 267, 488, 921	17	17	8	1	3400
		1, 595, 617, 2200	17	14	8	1	
		1, 1065, 2648, 2670	15	15	8	1	
		1, 1193, 2408, 2534	19	14	8	1	
		2, 391, 591, 3210	21	16	8	1	
19	3924	1, 332, 627, 1392	17	17	8	1	4050
		1, 370, 457, 1769	13	13	8	1	
		1, 614, 729, 2157	11	11	8	1	
		1, 1768, 3196, 3311	17	16	8	1	
		1, 2156, 3468, 3555	15	15	8	1	
20	4602	1, 47, 2688, 3171	19	19	9	1	4750
		1, 267, 2916, 3971	15	15	9	1	
		1, 632, 1687, 4336	27	14	9	1	
		1, 1394, 3025, 3772	19	19	9	1	
		2, 1114, 2679, 4431	17	14	9	1	
21	5412	1, 83, 1338, 4644	17	17	9	1	5580
		1, 83, 3714, 5142	19	18	9	1	
		1, 271, 1699, 5330	17	17	9	1	
		1, 373, 1886, 4615	19	19	9	1	
		1, 798, 3527, 5040	17	17	9	1	
22	6416	1, 826, 1219, 2100	13	13	9	1	6450
		1, 1082, 2246, 5039	17	17	9	1	
		1, 1173, 4611, 4838	17	17	9	1	
		1, 1378, 4171, 5335	21	21	9	1	
		1, 2260, 2595, 3082	19	19	9	1	

Table G.4: Directed circulant graphs of directed degree 5

Diameter k	Order	Generating set Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
2	19	1, 3, 12, 14, 15	3	3	1	1	21
3	40	1, 3, 7, 20, 29	5	4	1	2	56
		1, 3, 16, 25, 38	3	3	1	1	
		1, 3, 19, 26, 32	5	4	2	1	
		1, 6, 7, 22, 25	5	4	1	1	
		1, 6, 8, 33, 35	3	3	1	1	
		1, 6, 9, 14, 15	5	4	1	2	
		1, 6, 19, 24, 25	5	4	1	1	
		1, 9, 15, 22, 38	3	3	2	2	
		1, 12, 14, 18, 21	3	3	1	2	
4	88	1, 3, 49, 58, 72	5	5	2	1	126
		1, 7, 57, 62, 80	3	3	2	1	
		1, 11, 14, 30, 65	7	4	2	2	
		1, 18, 26, 55, 65	7	4	2	2	
5	168	1, 22, 119, 128, 135	5	5	3	1	252
		1, 34, 41, 50, 147	9	4	3	2	
6	273	1, 4, 68, 96, 119	5	5	3	1	462
		1, 5, 17, 112, 204	7	6	3	1	
		1, 17, 81, 155, 262	7	4	3	1	
7	447	1, 14, 171, 380, 426	5	5	3	1	792
		1, 17, 259, 381, 424	11	6	3	1	
		1, 21, 34, 128, 205	11	6	3	1	
		1, 22, 68, 277, 434	7	7	3	1	
		1, 24, 67, 189, 431	5	5	3	1	
		1, 37, 90, 312, 371	5	5	3	1	
8	689	1, 17, 26, 131, 433	7	7	3	1	792
		1, 27, 43, 303, 611	7	6	3	1	
		1, 40, 82, 277, 624	11	8	3	1	
9	1056	1, 65, 188, 604, 957	13	8	3	2	1080
		1, 100, 453, 869, 992	7	7	3	1	

G.2 Mixed circulant graphs of dimension 2

Table G.5: Mixed circulant graphs of directed degree 2, undirected degree 1

Diameter k	Order	Isomorphism class	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI
2	8	-	1, 3	3	3	1	2
3	12	-	1, 4	3	3	2	1
		-	1, 10	3	3	2	1
4	18	-	1, 4	7	4	2	1
		-	1, 16	3	3	2	1
5	24	-	1, 5	5	5	3	2
		-	1, 10	7	4	2	1
		-	1, 21	5	4	3	1
		-	1, 22	3	3	2	1
6	32	M2-1:0A	1, 7	5	5	3	1
		-	1, 5	5	5	3	1
		-	1, 10	5	5	3	1

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Table G.5: (cont.) Mixed circulant graphs of directed degree 2, undirected degree 1

Diameter k	Order	Isomorphism class	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI
7	42	-	1, 21	9	4	3	1
		-	1, 26	7	6	3	1
		M2-1:1A	1, 9	bipartite	6	4	1
		-	1, 13	bipartite	6	4	2
		-	1, 30	7	6	4	1
		-	1, 34	7	7	4	1
		-	2, 18	5	5	4	1
		-	2, 26	9	6	4	2
8	52	-	2, 39	5	5	4	1
		M2-1:2A	1, 8	13	6	4	1
		-	1, 11	7	7	4	1
		-	1, 16	7	7	4	1
		-	1, 34	5	5	4	1
		-	1, 37	7	7	4	1
9	66	-	1, 42	11	8	4	1
		M2-1:0A	1, 10	7	7	5	1
		-	1, 24	11	8	5	1
		-	1, 43	bipartite	6	5	2
		-	1, 57	bipartite	8	5	1
		-	2, 15	7	7	5	1
		-	2, 20	15	6	5	2
		-	2, 48	7	7	5	1
10	80	M2-1:1A	1, 12	11	8	6	1
		-	1, 29	9	9	6	1
		-	1, 52	7	7	6	1
11	94	M2-1:2A	1, 11	bipartite	8	6	1
		-	1, 14	9	9	6	1
		-	1, 18	9	9	6	1
		-	1, 30	7	7	6	1
		-	1, 34	13	10	6	1
		-	1, 37	bipartite	8	6	1
		-	1, 58	17	8	6	1
		-	1, 65	bipartite	10	6	1
		-	1, 84	9	9	6	1
		-	2, 22	7	7	6	1
		-	2, 28	11	8	6	1
		-	2, 36	9	9	6	1
12	112	M2-1:0A	1, 13	9	9	7	1
		-	1, 44	9	9	7	1
		-	1, 100	13	10	7	1
13	130	M2-1:1A	1, 15	bipartite	10	8	1
		-	1, 51	bipartite	10	8	2
		-	1, 80	13	10	8	1
		-	1, 116	11	11	8	1
		-	2, 30	9	9	8	1
		-	2, 95	9	9	8	1
		-	2, 102	15	10	8	2
14	148	M2-1:2A	1, 14	23	10	8	1
		-	1, 17	11	11	8	1
		-	1, 58	11	11	8	1
		-	1, 88	9	9	8	1
		-	1, 91	11	11	8	1

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Table G.5: (cont.) Mixed circulant graphs of directed degree 2, undirected degree 1

Diameter k	Order	Isomorphism class	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI
15	170	-	1, 132	17	12	8	1
		M2-1:0A	1, 16	11	11	9	1
		-	1, 70	17	12	9	1
		-	1, 101	bipartite	10	9	2
		-	1, 155	bipartite	12	9	1
		-	2, 32	25	10	9	2
		-	2, 55	11	11	9	1
		-	2, 140	11	11	9	1
16	192	M2-1:1A	1, 18	17	12	10	1
		-	1, 79	13	13	10	1
		-	1, 114	11	11	10	1

*plus the involution

Table G.6: Mixed circulant graphs of directed degree 1, undirected degree 2

Diameter k	Order	Isomorphism class	Generating set		Odd	Maximal	Aut
			Directed	Undirected	girth	levels	group CI
2	7	-	1	2	3	1	1
		-	1	3	3	1	1
3	13	M1-2:0	1	3	5	2	1
4	20	M1-2:1A	5	2	7	3	2
5	28	M1-2:2A	7	2	9	3	2
6	37	M1-2:0	1	5	9	4	1
7	48	M1-2:1A	8	3	11	5	2
8	60	M1-2:2A	10	3	13	5	2
9	73	M1-2:0	1	7	13	6	1
10	88	M1-2:1A	11	4	15	7	2
11	104	M1-2:2A	13	4	17	7	2
12	121	M1-2:0	1	9	17	8	1
13	140	M1-2:1A	14	5	19	9	2
14	160	M1-2:2A	16	5	21	9	2
15	181	M1-2:0	1	11	21	10	1
16	204	M1-2:1A	17	6	23	11	2

Table G.7: Mixed circulant graphs of directed degree 1, undirected degree 3

Diameter k	Order	Isomorphism class	Generating set*		Odd girth	Maximal levels	Aut group CI
			Directed	Undirected			
2	10	-	1	2	3	1	1
		-	2	1	3	1	1
3	20	M1-3:0	1	3	5	2	1
4	32	M1-3:1	1	3	7	2	1
5	44	M1-3:2A	1	3	9	2	1
		M1-3:2B	1	5	7	3	1
		-	1	4	5	3	1
		-	1	18	7	3	1
6	64	M1-3:0	1	5	9	4	1
7	84	M1-3:1	1	5	11	4	1
8	104	M1-3:2A	1	5	13	4	1
		M1-3:2B	1	7	11	5	1
9	132	M1-3:0	1	7	13	6	1
10	160	M1-3:1	1	7	15	6	1
11	188	M1-3:2A	1	7	17	6	1
		M1-3:2B	1	9	15	7	1
12	224	M1-3:0	1	9	17	8	1
13	260	M1-3:1	1	9	19	8	1
14	296	M1-3:2A	1	9	21	8	1
		M1-3:2B	1	11	19	9	1
15	340	M1-3:0	1	11	21	10	1
16	384	M1-3:1	1	11	23	10	1

*plus the involution

G.3 Mixed circulant graphs of dimension 3**Table G.8:** Mixed circulant graphs of directed degree 3, undirected degree 1

Diameter k	Order	Generating set*	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
		Directed					
2	12	1, 3, 5	3	3	1	2	14
3	24	1, 3, 17	5	4	2	2	30
		1, 15, 17	5	4	2	2	
4	40	1, 9, 15	5	4	2	2	54
		1, 9, 35	5	4	2	2	
		1, 15, 17	5	5	2	1	
		1, 17, 35	5	5	2	1	
5	62	1, 6, 14	5	5	3	1	90
		1, 6, 45	5	5	3	1	
		1, 20, 27	5	5	3	1	
		1, 26, 54	5	5	3	1	
6	92	1, 7, 29	7	7	3	1	140
		1, 27, 29	7	7	2	1	
		1, 27, 51	9	6	3	1	
7	132	1, 31, 47	7	7	3	1	204
8	176	1, 25, 55	9	8	4	1	284
		1, 25, 143	9	8	4	1	

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Table G.8: (cont.) Mixed circulant graphs of directed degree 3, undirected degree 1

Diameter k	Order	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
		1, 55, 81	9	9	4	1	
		1, 81, 143	9	9	4	1	
9	234	1, 52, 226	9	9	4	1	384
		1, 169, 190	9	9	4	1	
10	300	1, 177, 235	11	11	5	1	506
11	380	1, 67, 165	11	11	6	1	650
		1, 173, 235	13	10	6	1	
12	462	1, 250, 280	9	9	5	1	818
		1, 304, 343	15	10	5	1	
13	576	1, 110, 455	13	13	6	1	1014
		1, 119, 130	13	13	6	1	
14	688	1, 13, 249	13	12	6	1	1240
		1, 105, 333	13	12	6	1	
		1, 131, 199	13	13	7	1	
		1, 131, 543	13	13	7	1	
		1, 199, 475	13	13	7	1	
		1, 223, 661	13	13	7	1	
15	822	1, 31, 484	15	15	7	1	1496
		1, 109, 744	13	13	8	1	
		1, 220, 670	19	16	7	1	
		1, 267, 592	13	13	8	1	
		1, 520, 744	17	14	8	1	
		2, 123, 362	17	14	8	1	
16	978	1, 188, 285	15	15	8	1784	
		1, 188, 774	15	15	8	1	
		1, 207, 476	15	15	8	1	
		1, 476, 696	15	15	8	1	
17	1140	1, 219, 332	13	13	9	1	2108
		1, 219, 902	17	14	9	1	
18	1328	1, 53, 91	17	16	8	1	2470
		1, 53, 755	17	16	8	1	
		1, 91, 717	17	16	8	1	
		1, 127, 213	17	16	8	1	
19	1554	1, 127, 1216	15	15	10	1	2870
		1, 187, 1450	15	15	10	1	
		1, 904, 1216	19	16	10	1	
		1, 1018, 1354	19	16	10	1	
20	1776	1, 76, 1526	19	19	10	1	3310
		1, 638, 964	19	19	10	1	
21	1992	1, 371, 549	21	21	10	1	3794
		1, 371, 1545	27	20	10	1	
		1, 549, 1367	21	21	10	1	
		1, 647, 1623	27	20	10	1	
22	2286	1, 310, 973	19	19	11	1	4324
		1, 316, 1948	19	19	11	1	
		1, 601, 700	19	19	11	1	
		1, 700, 1744	19	19	11	1	
23	2610	1, 246, 1030	25	22	9	1	4900
		1, 246, 2335	27	22	9	1	
		1, 276, 1060	19	19	9	1	

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Table G.8: (cont.) Mixed circulant graphs of directed degree 3, undirected degree 1

Diameter k	Order	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
24	2892	1, 276, 2365	19	19	9	1	5524
		1, 1030, 1551	25	22	9	1	
		1, 1060, 1581	27	20	9	1	
		1, 1551, 2335	bipartite	22	9	1	
		1, 1581, 2365	bipartite	20	9	1	
		2, 492, 755	21	21	9	1	
		2, 492, 2060	21	21	9	1	
		2, 552, 815	19	19	9	1	
		2, 552, 2120	19	19	9	1	
		2, 755, 1797	21	21	9	1	
		2, 815, 1857	21	20	9	1	
		2, 1797, 2060	21	21	9	1	
		2, 1857, 2120	21	20	9	1	
		1, 425, 570	23	18	12	1	
		1, 425, 2016	23	18	12	1	
25	3232	1, 56, 257	29	24	10	1	6200
		1, 56, 1873	23	23	10	1	
		1, 257, 1672	23	23	10	1	
		1, 416, 617	21	21	10	1	
		1, 416, 2233	29	20	10	1	
		1, 617, 2032	21	21	10	1	
		1, 1000, 1201	25	24	10	1	
		1, 1360, 3177	29	20	10	1	
		1, 171, 1339	25	25	12	1	6930
26	3616	1, 171, 3147	29	24	12	1	
		1, 177, 2675	25	25	12	1	
		1, 1339, 1979	29	24	12	1	
27	4040	1, 1076, 3691	31	26	13	1	7714
		1, 1496, 2211	25	25	13	1	
28	4464	1, 211, 1653	27	27	14	1	8554
		1, 211, 3885	33	26	14	1	
		1, 1383, 1555	27	27	14	1	
		1, 1555, 3615	33	26	14	1	
29	4884	1, 964, 1372	27	24	11	1	8554
		1, 964, 3814	25	25	11	1	
		1, 1071, 1479	31	26	11	1	
		1, 1071, 3921	27	27	11	1	
		1, 1372, 3406	25	25	11	1	
		1, 1479, 3513	33	26	11	1	
		1, 3406, 3814	33	24	11	1	
		1, 3513, 3921	27	27	11	1	
		2, 285, 1752	29	24	11	1	
		2, 285, 4194	23	23	11	1	
		2, 1185, 1590	31	26	11	1	
		2, 1185, 4032	31	26	11	1	
		3, 1278, 1688	23	23	11	1	
		3, 1688, 3720	29	24	11	1	
		3, 2106, 3188	25	25	11	1	
		3, 3188, 4548	25	25	11	1	
30	5386	1, 39, 942	35	26	12	1	8554
		1, 39, 3635	bipartite	26	12	1	
		1, 161, 1676	29	29	12	1	

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Table G.8: (cont.) Mixed circulant graphs of directed degree 3, undirected degree 1

Diameter k	Order	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
		1, 161, 4369	bipartite	28	12	1	
		1, 184, 1382	29	29	12	1	
		1, 184, 4075	31	28	12	1	
		1, 243, 1398	25	25	12	1	
		1, 373, 2259	bipartite	24	12	1	
		1, 373, 4952	23	23	12	1	
		1, 435, 2321	bipartite	26	12	1	
		1, 435, 5014	29	26	12	1	
		1, 638, 4008	29	29	15	1	
		1, 942, 2732	25	25	12	1	
		1, 1018, 2533	23	23	12	1	
		1, 1018, 5226	33	24	12	1	
		1, 1063, 4206	27	27	12	1	
		1, 1181, 4324	29	29	12	1	
		1, 1296, 2451	23	23	12	1	
		1, 1296, 5144	33	24	12	1	
		1, 1312, 2510	31	26	12	1	
		1, 1312, 5203	31	26	12	1	
		1, 1398, 2936	25	25	12	1	
		1, 1513, 3756	27	27	12	1	
		1, 1676, 2854	35	28	12	1	
		1, 1752, 5348	27	27	12	1	
		1, 1929, 2522	33	28	15	1	
		1, 2630, 5044	43	28	15	1	
		1, 3066, 4952	39	24	12	1	
		1, 3128, 5014	27	27	12	1	
		1, 3756, 4206	29	26	12	1	
		1, 3874, 4324	29	29	12	1	
		1, 4622, 5215	29	29	15	1	
		2, 78, 1884	25	25	12	1	
		2, 322, 3352	31	28	12	1	
		2, 746, 4518	23	23	12	1	
		2, 870, 4642	35	26	12	1	
31	5874	1, 885, 2166	27	27	15	1	8554
		1, 3822, 5103	33	28	15	1	
		2, 1395, 1770	37	28	15	1	
		2, 4332, 4707	27	27	15	1	
32	6440	7, 30, 2277	31	28	13	1	8554
		7, 30, 5497	37	28	13	1	
		7, 437, 1410	27	27	13	1	
		7, 897, 5090	27	27	13	1	
33	7182	1, 1519, 2031	bipartite	30	13	1	8554
		1, 1519, 5622	29	29	13	1	
		1, 1561, 2073	bipartite	28	13	1	
		1, 1561, 5664	37	28	13	1	
		1, 2031, 5110	37	30	13	1	
		1, 2073, 5152	31	28	13	1	
		1, 5110, 5622	29	29	13	1	
		1, 5152, 5664	31	28	13	1	
		2, 471, 3038	33	30	13	1	
		2, 471, 6629	33	30	13	1	
		2, 555, 3122	27	27	13	1	
		2, 555, 6713	27	27	13	1	

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Table G.8: (cont.) Mixed circulant graphs of directed degree 3, undirected degree 1

Diameter k	Order	Generating set* Directed	Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
34	7700	2, 3038, 4062	29	29	13	1	8554
		2, 3122, 4146	27	27	13	1	
		2, 4062, 6629	29	29	13	1	
		2, 4146, 6713	27	27	13	1	
		2, 475, 1177	29	29	13	1	
		2, 475, 5027	29	29	13	1	
		2, 777, 1325	41	30	13	1	
		2, 777, 5175	37	30	13	1	
		2, 2527, 3075	29	29	13	1	
		2, 2527, 6925	33	28	13	1	
		2, 2675, 3377	29	29	13	1	
		2, 2675, 7227	35	28	13	1	
		4, 375, 3129	33	28	13	1	
		4, 375, 6979	29	29	13	1	
		4, 429, 2875	33	30	13	1	
		4, 429, 6725	33	30	13	1	
		4, 725, 3479	29	29	13	1	
		4, 725, 7329	29	29	13	1	
		4, 979, 3425	29	29	13	1	
		4, 979, 7275	39	28	13	1	
35	8382	1, 748, 2652	31	31	14	1	8554
		1, 748, 6843	35	32	14	1	
		1, 1540, 3444	47	28	14	1	
		1, 1540, 7635	29	29	14	1	
		1, 2652, 4939	31	31	14	1	
		1, 3444, 5731	29	29	14	1	
		1, 4939, 6843	bipartite	32	14	1	
		1, 5731, 7635	bipartite	28	14	1	
		2, 1113, 1496	35	32	14	1	
		2, 1113, 5687	39	32	14	1	
		2, 1496, 5304	31	31	14	1	
		2, 2697, 3080	29	29	14	1	
		2, 2697, 7271	39	28	14	1	
		2, 3080, 6888	39	28	14	1	
		2, 5304, 5687	31	31	14	1	
		2, 6888, 7271	29	29	14	1	
36	9000	1, 2284, 3506	35	35	18	1	9150
		1, 6784, 8006	35	35	18	1	
37	9742	1, 238, 1113	33	33	20	1	9900
		1, 238, 5984	33	33	20	1	
		1, 307, 721	bipartite	34	20	1	
		1, 307, 5592	45	34	20	1	
		1, 372, 858	33	33	20	1	
		1, 372, 5729	41	34	20	1	
		1, 781, 3437	bipartite	30	20	1	
		1, 781, 8308	41	30	20	1	
		1, 853, 2768	45	34	20	1	
		1, 853, 7639	bipartite	34	20	1	
		1, 1117, 3176	31	31	20	1	
		1, 1435, 4091	bipartite	32	20	1	
		1, 1435, 8962	41	32	20	1	
		1, 1696, 3755	33	33	20	1	
		1, 1696, 8626	33	33	20	1	

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Table G.8: (cont.) Mixed circulant graphs of directed degree 3, undirected degree 1

Diameter k	Order	Generating set*		Odd girth	Girth	Maximal levels	Aut group CI	Checked up to
		Directed						
		1, 2104, 8890		41	32	20	1	
		1, 2768, 5724		41	34	20	1	
		1, 3176, 5988		41	30	20	1	
		1, 3437, 5652		31	31	20	1	
		1, 4014, 4500		33	33	20	1	
		1, 4091, 6306		31	31	20	1	
		1, 4151, 9436		49	32	20	1	
		1, 4500, 8885		41	34	20	1	
		1, 4634, 8630		31	31	20	1	
		1, 5178, 5592		41	34	20	1	
		1, 5652, 8308		31	31	20	1	
		1, 6306, 8962		31	31	20	1	
		1, 9022, 9436		31	31	20	1	
		2, 476, 2226		33	33	20	1	
		2, 1562, 6874		47	30	20	1	
		2, 1706, 5536		33	33	20	1	
		2, 2870, 8182		31	31	20	1	

*plus the involution

Table G.9: Mixed circulant graphs of directed degree 2, undirected degree 2

Diameter k	Order	Generating set		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
2	13	1, 12	5	5	1	4	14
3	25	1, 24	7	7	1	4	30
4	41	1, 16	18	5	2	1	55
5	66	2, 55	7	9	2	1	91
6	95	1, 11	15	9	3	1	140
		1, 56	35	11	3	2	
7	136	1, 67	42	13	3	2	204
8	182	2, 65	77	9	3	1	285
9	241	1, 128	67	9	5	1	385
10	314	2, 218	65	11	5	1	506
11	391	1, 171	126	15	6	1	650
12	489	1, 80	192	11	6	1	819
13	609	1, 202	159	21	5	2	1015
14	717	1, 70	190	15	8	1	1240
15	855	1, 47	64	11	7	1	1496
16	1024	1, 818	28	11	7	1	1785
17	1206	1, 780	523	19	9	1	2109
18	1381	1, 664	536	17	9	1	2470
		1, 938	84	17	9	1	
19	1609	1, 417	709	21	8	1	2870
		1, 1200	361	21	8	1	
20	1836	1, 576	415	15	10	1	3311
21	2093	1, 842	463	15	9	1	3795
		1, 1793	763	23	9	2	

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Table G.9: (cont.) Mixed circulant graphs of directed degree 2, undirected degree 2

Diameter k	Order	Generating set		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
		1, 1846	537	17	11	1	
22	2395	1, 1436	775	33	9	2	4324
23	2709	1, 386	1001	33	13	2	4900
24	3056	1, 1145	520	27	10	2	5525
25	3380	1, 116	279	25	14	1	6201
26	3801	1, 474	258	17	11	1	6930
27	4203	1, 3807	1043	17	15	1	7714
28	4663	1, 447	613	29	15	1	8555
29	5135	1, 339	1736	31	12	1	8850
30	5603	1, 2311	626	43	12	1	8850
31	6193	1, 3073	225	21	13	1	8850
32	6769	1, 1933	406	47	13	2	8850
33	7441	1, 5314	406	49	13	2	8850
34	8041	1, 5116	2387	37	14	2	8850
35	8655	1, 88	711	51	14	1	8850
36	9432	3, 4284	232	23	15	1	9600
37	10192	1, 1273	152	57	15	2	10350
38	10938	1, 10536	4411	23	21	1	11160
39	11819	1, 4039	1205	41	16	1	12000
40	12631	1, 9320	1217	57	16	1	12900
41	13585	5, 969	1887	27	17	1	13800
42	14493	1, 2665	5673	43	17	1	14780
43	15705	1, 12214	5751	63	17	2	15800
44	16688	1, 1191	1442	47	18	2	16850
45	17693	1, 8165	1053	61	25	2	17950

Table G.10: Mixed circulant graphs of directed degree 2, undirected degree 3

Diameter k	Order	Generating set*		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
2	14	1, 2	3	3	1	1	18
		1, 4	2	3	1	1	
		1, 10	2	3	1	1	
		1, 12	3	3	1	1	
3	32	1, 3	5	5	1	1	44
		1, 3	7	5	2	1	
		1, 4	9	5	2	1	
		1, 20	9	5	2	1	
		2, 12	3	5	2	1	
		2, 28	5	3	2	1	
4	60	1, 3	7	7	2	1	84
		1, 7	9	5	2	1	
		1, 11	7	5	2	1	
		1, 55	9	7	2	1	
5	98	1, 22	45	9	2	1	146

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Table G.10: (cont.) Mixed circulant graphs of directed degree 2, undirected degree 3

Diameter k	Order	Generating set*		Odd girth	Maximal levels	Aut group CI	Checked up to
		1, 78	18	5	2	1	
6	160	1, 31	35	11	3	2	230
7	220	1, 11	15	11	3	1	344
		1, 131	85	13	3	2	
8	312	1, 299	23	11	4	1	488
		4, 256	9	7	4	1	
9	404	1, 31	95	13	4	1	670
10	544	1, 177	11	17	4	1	890
11	684	1, 379	45	11	5	2	1156
12	864	1, 155	105	11	6	1	1468
13	1068	2, 710	123	21	5	2	1834
		4, 352	21	23	5	2	
14	1320	5, 875	159	23	5	2	2254
15	1564	1, 1123	319	17	7	1	2736
16	1848	1, 549	901	21	8	1	3280
		3, 195	223	17	8	1	
17	2172	1, 427	17	25	7	1	3894
18	2560	1, 2007	227	17	9	1	4578
19	2940	1, 2387	967	17	9	1	5340
20	3392	1, 685	1617	17	10	1	6180
21	3864	1, 2167	945	31	8	1	7106
22	4480	1, 1791	545	35	9	2	8120
23	5080	2, 4062	605	35	9	2	8120
		2, 4062	665	37	9	2	
24	5760	1, 1151	1405	37	9	2	8120
25	6304	1, 1969	1544	17	11	1	8120
26	7048	1, 861	2799	29	13	1	8120
27	7806	1, 3414	2751	39	11	1	8120
28	8736	1, 4581	1926	29	12	1	8980
		1, 4581	2442	19	12	1	
29	9598	1, 3162	1900	29	16	1	9900
30	10560	5, 5885	441	29	14	1	10850
31	11640	3, 4615	2597	45	12	1	11900
32	12880	1, 9199	4739	49	13	2	13000
33	14084	2, 12070	3157	49	13	2	14200
34	15400	1, 2199	1351	51	13	2	15500
35	16440	1, 821	3190	21	19	1	16850
36	17856	1, 4969	4047	53	14	1	18300
37	19300	1, 12351	8025	45	19	2	19700

*plus the involution

Table G.11: Mixed circulant graphs of directed degree 1, undirected degree 4

Diameter		Generating set		Odd	Maximal	Aut	Checked
k	Order	Directed	Undirected	girth	levels	group CI	up to
2	15	5	1, 4	3	1	2	19
		2	1, 6	3	1	1	
		5	1, 6	3	1	1	
3	32	14	1, 5	5	1	1	44
4	63	21	1, 8	3	2	2	85
5	106	16	1, 48	9	2	1	146
6	164	41	2, 18	11	3	4	231
7	244	61	2, 22	13	3	4	344
8	340	85	2, 26	15	3	4	489
9	458	136	1, 222	15	4	1	670
10	602	241	1, 188	17	4	1	891
11	779	320	1, 361	17	5	1	1156
12	981	156	1, 46	19	5	1	1469
13	1219	174	1, 64	21	6	1	1834
14	1491	354	1, 504	23	6	1	2255
15	1807	54	1, 433	23	7	1	2736
16	2157	766	1, 342	25	7	1	3281
17	2544	778	1, 630	27	8	1	3894
18	2984	663	1, 574	29	8	1	4579
19	3479	1706	1, 336	29	9	1	5340
20	4017	1886	1, 1224	31	9	1	6181
21	4595	1875	1, 369	33	9	1	7106
22	5237	191	1, 2101	35	10	1	8119
23	5951	1957	1, 321	35	11	1	9224
24	6717	3027	1, 2738	37	11	1	10425
25	7531	2121	1, 2858	39	11	1	11726
26	8401	3378	1, 3113	39	13	1	12000
27	9379	4158	1, 1939	41	13	1	12000
28	10413	4079	1, 5040	43	13	1	12000
29	11503	2490	1, 350	45	13	1	12000
30	12649	3577	1, 517	47	13	1	12800
31	13919	4165	1, 2954	47	15	1	14020
32	15261	5287	1, 6975	49	15	1	15360
33	16667	904	1, 2202	51	15	1	16770
34	18137	7873	1, 990	53	15	1	18240
35	19727	3268	1, 358	53	17	1	19840
36	21417	4467	1, 5261	55	17	1	21520
37	23179	9385	1, 6678	57	17	1	23300
38	25013	2916	1, 2461	59	17	1	25100
39	26959	11244	1, 9207	59	19	1	27060
40	29037	8171	1, 10803	61	19	1	29140
41	31195	13795	1, 4854	63	19	1	31300
42	33433	7896	1, 9246	65	19	1	33550
43	35771	13396	1, 8543	65	21	1	35900
44	38277	9847	1, 13029	67	21	1	38400

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Table G.11: (cont.) Mixed circulant graphs of directed degree 1, undirected degree 4

Diameter k	Order	Generating set		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
45	40871	2355	1, 10938	69	21	1	41000
46	43553	4122	1, 3579	71	21	1	43650
47	46323	20182	1, 19227	73	21	1	46450
48	49293	5970	1, 4600	73	23	1	49400
49	52363	12864	1, 5441	75	23	1	52500

Table G.12: Mixed circulant graphs of directed degree 1, undirected degree 5

Diameter k	Order	Generating set*		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
2	20	3	1, 4	3	1	1	24
		3	1, 6	3	1	1	
3	46	20	1, 14	5	2	1	62
		15	1, 18	5	2	1	
4	88	31	1, 5	7	2	1	128
		27	1, 7	7	2	1	
5	164	57	1, 7	9	2	1	230
6	264	91	1, 9	11	2	1	376
7	392	173	1, 91	11	3	1	574
		55	1, 173	11	3	1	
8	564	249	1, 131	13	3	1	832
		79	1, 249	13	3	1	
9	772	157	1, 13	15	4	1	1158
10	1040	211	1, 15	17	4	1	1560
11	1348	273	1, 17	19	4	1	2046
12	1724	529	1, 273	19	5	1	2624
13	2152	781	1, 381	21	5	1	3302
14	2648	381	1, 19	23	6	1	4088
15	3232	1545	1, 569	23	7	1	4990
16	3900	847	1, 1025	25	7	1	6016
17	4632	2013	1, 2207	27	7	1	7174
18	5456	203	1, 1827	29	8	1	8472
19	6372	1887	1, 2133	31	8	1	9918
20	7396	2887	1, 715	31	9	1	11520
21	8512	4153	1, 1031	33	9	1	13286
22	9708	3079	1, 1665	35	9	1	13300
23	11040	4705	1, 1261	35	11	1	13300
24	12524	425	1, 5101	37	11	1	13300
25	14104	2391	1, 485	39	11	1	14300
26	15780	3881	1, 769	41	11	1	16000
27	17604	8125	1, 1903	43	12	1	17720
28	19596	2767	1, 455	43	13	1	19700
29	21720	7851	1, 1315	45	13	1	21900
30	23956	8947	1, 5477	47	13	1	24100
31	26356	7165	1, 2053	47	14	1	26500
32	28924	5733	1, 4957	49	15	1	29100
33	31672	2523	1, 8695	51	15	1	31800
34	34548	611	1, 9777	53	15	1	34700
35	37552	10967	1, 12289	55	15	1	37700
36	40820	18019	5, 2223	55	17	1	40950
37	44272	17337	1, 2161	57	17	1	44400

*plus the involution

G.4 Mixed circulant graphs of dimension 4

Table G.13: Mixed circulant graphs of directed degree 3, undirected degree 2

Diameter k	Order	Generating set		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
2	17	1, 4, 10	5	3	1	1	20
3	39	1, 12, 14	18	5	2	2	50
4	75	1, 24, 26	18	7	2	2	105
5	131	1, 40, 42	32	7	2	1	196
6	206	1, 24, 135	42	7	3	1	336
7	318	1, 162, 309	130	9	3	1	540
8	465	1, 50, 354	42	9	4	1	825
9	660	4, 36, 612	67	5	4	1	1210
10	902	1, 508, 696	153	13	5	1	1716
11	1198	1, 341, 792	152	13	5	1	1800
12	1611	1, 460, 600	772	9	6	1	1800
13	2043	1, 655, 990	389	17	6	1	2160
14	2575	1, 185, 232	407	13	6	1	2750
15	3212	1, 1182, 1589	990	17	7	1	3350
16	3959	1, 409, 3667	982	17	7	1	4120
17	4913	1, 338, 3814	414	13	7	1	5000

Table G.14: Mixed circulant graphs of directed degree 2, undirected degree 4

Diameter k	Order	Generating set		Odd girth	Maximal levels	Aut group CI	Checked up to
		Directed	Undirected				
2	20	1, 3	4, 9	3	1	1	26
		1, 10	2, 7	3	1	1	
		1, 10	7, 8	3	1	1	
		4, 8	3, 5	3	1	1	
3	51	1, 5	7, 18	5	2	1	70
		3, 17	21, 23	3	2	1	
4	103	1, 27	5, 29	5	2	1	155
5	200	25, 75	4, 28	9	2	8	301
6	344	1, 171	76, 142	11	3	2	532
7	514	1, 228	7, 63	9	3	1	876
8	788	1, 393	58, 126	13	3	2	1365
9	1160	1, 579	46, 56	15	3	2	2035
10	1596	1, 797	360, 700	15	3	2	2926
11	2206	1, 1102	211, 323	19	3	1	3100
		1, 1102	239, 899	19	3	1	
12	2934	1, 1466	125, 1057	21	3	1	3100
		1, 1466	473, 751	21	3	1	
13	3778	1, 1888	161, 1361	23	3	1	4000
		1, 1888	609, 967	23	3	1	
14	4962	2, 1652	960, 2229	23	5	2	5100
15	6240	1, 4159	552, 3084	23	5	2	6400

Table G.15: Mixed circulant graphs of directed degree 1, undirected degree 6

Diameter k		Order	Generating set Directed Undirected	Odd girth	Maximal levels	Aut group CI	Checked up to
2	27	11	1, 4, 10	3	1	1	33
3	70	23	1, 7, 18	5	2	1	96
4	155	31	1, 36, 56	5	2	3	225
5	282	130	1, 30, 108	7	2	1	456
6	528	176	1, 144, 199	3	2	2	833
7	869	290	1, 40, 321	13	2	1	1408
8	1323	46	1, 138, 148	15	2	1	2241
		639	1, 594, 602	15	2	1	
9	2020	210	1, 118, 841	15	3	1	3400
10	2896	724	1, 49, 645	17	3	1	3400
11	4024	906	1, 401, 1696	19	3	1	4200
12	5627	1536	1, 751, 1301	19	4	1	5750
13	7433	3305	1, 54, 1714	21	4	1	7600
14	9663	4124	1, 742, 2035	23	4	1	9900